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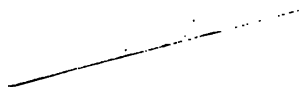
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HEAD MASTER OF STATIONERS' SCHOOL, LONDON.



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## PREFACE.

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THE present edition of Euclid's Elements of Geometry, designed chiefly for the higher classes in schools and colleges, contains the parts usually read in the Universities, and is arranged on the same general plan which has proved so successful in the edition of the Four Books published, for the use of younger students, under the name of The School Euclid. Like many of its predecessors, it is based on the work of Dr. Simson, whose valuable edition has been generally followed. The alterations introduced have been chiefly confined to the arrangement, in which, however, considerable changes and, it is hoped, improvements have been made. Among these the following may be noticed:—

1. The references to previous propositions and definitions, generally given in the common editions in the margin, are here collected immediately after the enunciations. An opportunity is thus afforded to the student of revising and making himself thoroughly master, before he begins the demonstration, of the references of which he is about to make use, so that, when they occur in their proper place in the text, their application may be at once perceived, without necessitating any pause in the reasoning in order to make sure of the reference before advancing to the next step in the proof.

2. In describing the figures, the parts which are given in the enunciation are represented by dark lines, and those which are added in the course of the demonstration by dotted lines. The process of the construction is thus exhibited to the eye, and the *data* and the *quæsitæ* of the problem can always be distinguished at a glance.

3. In the demonstrations, the several steps of the proof are arranged in a logical form, by giving the premisses and the conclusion always in separate lines, and in a different type; the 'construction' and the 'demonstration' are distinguished by separate headings; and, as a further aid to the student, the enunciations are broken into paragraphs, and the demonstrations into corresponding divisions, wherever the proposition consists of more than one case.

In this way the constituent parts of a proposition are presented separately, part by part; and the learner, knowing exactly where one begins and the other ends, is enabled to make himself master of the one before he proceeds to the other.

The editions of Hill, Blakelock, and Williams were among the earliest to show the advantages of printing separately the parts of a proposition and its demonstration; and they have been followed with great success by several modern editors. The plan adopted by them, of printing every sentence, or part of a sentence, which contains a new step in the reasoning, in a separate 'line,\*' has been followed in the

\* The method of exhibiting the propositions in separate lines, now so extensively adopted, was first introduced in an edition of the 'Elements' published, in 1726, by the Rev. Henry Hill, who thus refers to it in his Preface :—

'The Method I have taken in writing is entirely new; wholly different from any that has hitherto been made choice of by any of the interpreters of Euclid, and such as does not so much require intent and severe thinking as a bare and easie inspection.

'The better to avoid obscurity and confusion, and that painful attention which we have a general aversion to, and which disturbs and distracts the mind, I have set each and every demonstration out in lines or steps; so that the several mediums that are necessary in order to investigate the truth of many propositions, might no ways puzzle or perplex the young geometrician; but that he may have a clear and perfect notion and idea of the most intricate Demonstrations in the Elements of Geometry.'—*The First Six, together with the Eleventh and Twelfth Books of Euclid's Elements, demonstrated after a new, plain, and easie Method*, by Henry Hill, A.M. London: 1726.

present edition, but to a greater extent, and with increased distinctness. To avoid the confused appearance produced by the lines being scattered irregularly over the page, as in previous texts of the 'Elements' on this plan, the lines have been printed so as to commence uniformly from the side of the page. Every conclusion is what printers technically term 'indented,' and the applicable part of it—that is, the part made use of or referred to in the subsequent reasoning—is, to mark its importance, printed in italics; the final conclusion, and the thing to be proved or to be done, standing out prominently from the rest of the page in bold 'Clarendon' type, so as to keep constantly before the eye the object towards which the whole process of the reasoning is directed.

The notes appended to the text are necessarily brief, but it is hoped that they will be found to touch upon most of the points of interest connected with the definitions and propositions. The notes are followed by a classified index of the propositions of each Book, in which all the theorems relating to the same subject are, for the convenience of reference and comparison, collected in one view.

In the Appendix the enunciations and corollaries, together with questions on the definitions, axioms, &c. of each Book are printed separately, to assist teachers in reviewing their classes, and to aid the student in the work of self-examination. Algebraical and arithmetical solutions of the propositions of the Second and Fifth Books have also been given, and to the whole is appended a large collection of geometrical exercises for solution. For the convenience of those using other editions of 'Euclid,' the Appendix has been published in a separate form.

A. K. ISBISTER.

LONDON: *January 3, 1865.*





THE  
ELEMENTS OF EUCLID.

---

BOOK I.

DEFINITIONS.

I.

A POINT is that which hath no parts, or which hath no magnitude.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A straight line is that which lies evenly between its extreme points.

V.

A superficies is that which hath only length and breadth.

VI.

The extremities of a superficies are lines.

VII.

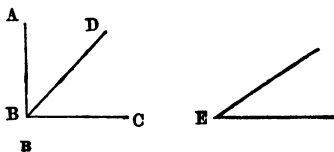
A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

VIII.

A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

IX.

A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.



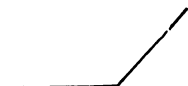
N.B. — When several angles are at one point B, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines containing the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of those straight lines, and the other upon the other line: Thus the angle which is contained by the straight lines, AB, CB, is named the angle ABC, or CBA; that which is contained by AB, DB, is named the angle ABD, or DBA; and that which is contained by DB, CB, is called the angle DBC, or CBD; but, if there be only one angle at a point, it may be expressed by a letter placed at that point: as the angle at E.

## X.



When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.

## XI.



An obtuse angle is that which is greater than a right angle.

## XII.



An acute angle is that which is less than a right angle.

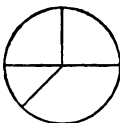
## XIII.

A term or boundary is the extremity of any thing.

## XIV.

A figure is that which is enclosed by one or more boundaries.

## XV.

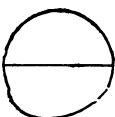


A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

## XVI.

And this point is called the centre of the circle.

## XVII.



A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

## XVIII.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

## XIX.

A segment of a circle is the figure contained by a straight line, and the circumference it cuts off.



## XX.

Rectilineal figures are those which are contained by straight lines.

## XXI.

Trilateral figures, or triangles, by three straight lines.

## XXII.

Quadrilateral, by four straight lines.

## XXIII.

Multilateral figures, or polygons, by more than four straight lines.

## XXIV.

Of three-sided figures, an equilateral triangle is that which has three equal sides.



## XXV.

An isosceles triangle, is that which has only two sides equal.



## XXVI.

A scalene triangle, is that which has three unequal sides.



## XXVII.

A right-angled triangle, is that which has a right angle.



## XXVIII.

An obtuse-angled triangle, is that which has an obtuse angle.



## XXIX.

An acute-angled triangle, is that which has three acute angles.



## XXX.



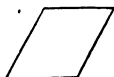
Of four-sided figures, a square is that which has all its sides equal, and all its angles right angles.

## XXXI.



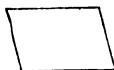
An oblong, is that which has all its angles right angles, but has not all its sides equal.

## XXXII.



A rhombus, is that which has all its sides equal, but its angles are not right angles.

## XXXIII.



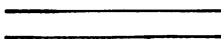
A rhomboid, is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

## XXXIV.

All other four-sided figures besides these, are called Trapeziums.

## XXXV.

Parallel straight lines, are such as are in the same plane, and which being produced ever so far both ways do not meet.



## POSTULATES.

## I.

Let it be granted that a straight line may be drawn from any one point to any other point.

## II.

That a terminated straight line may be produced to any length in a straight line.

## III.

And that a circle may be described from any centre, at any distance from that centre.

## AXIOMS.

## I.

Things which are equal to the same thing, are equal to one another.

## II.

If equals be added to equals, the wholes are equal.

## III.

If equals be taken from equals, the remainders are equal.

## IV.

If equals be added to unequals, the wholes are unequal.

## V.

If equals be taken from unequals, the remainders are unequal.

## VI.

Things which are double of the same, are equal to one another.

## VII.

Things which are halves of the same, are equal to one another.

## VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

## IX.

The whole is greater than its part.

## X.

Two straight lines cannot inclose a space.

## XI.

All right angles are equal to one another.

## XII.

‘ If a straight line meets two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.’

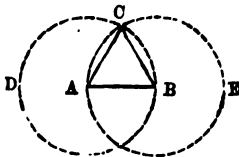
## PROP. I.—PROBLEM.

*To describe an equilateral triangle upon a given finite straight line.*

\*(References—Def. 15; ax. 1; post. 1, 3.)

Let  $AB$  be the given straight line.

It is required to describe on  $AB$  an equilateral triangle.



## CONSTRUCTION

From the centre  $A$ , at the distance  $AB$ , describe the circle  $BCD$ ;  
(post. 3)

from the centre  $B$ , at the distance  $BA$ , describe the circle  $ACE$ ;  
and from the point  $C$ , in which the circles cut one another, draw the  
straight lines  $CA$ ,  $CB$ , to the points  $A$ ,  $B$ . (post. 1.)

**Then  $ABC$  shall be an equilateral triangle.**

## DEMONSTRATION

Because the point  $A$  is the centre of the circle  $BCD$ ,

*therefore  $AC$  is equal to  $AB$ ;* (def. 15)

and because the point  $B$  is the centre of the circle  $ACE$ ,

*therefore  $BC$  is equal to  $BA$ .*

But it has been proved that  $CA$  is equal to  $AB$ ;

*therefore  $CA$ ,  $CB$  are each of them equal to  $AB$ ;*

but things which are equal to the same thing are equal to one another;  
(ax. 1)

*therefore  $CA$  is equal to  $CB$ ;*

wherefore  $CA$ ,  $AB$ ,  $BC$ , are equal to one another;

**and the triangle  $ABC$  is therefore equilateral;**

and it is described upon the given straight line  $AB$ .

Which was required to be done.

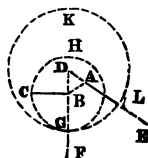
## PROP. II.—PROBLEM.

*From a given point to draw a straight line equal to a given straight line.*

(References—Prop. I. 1; post. 1, 2, 3; ax. 1, 3; def. 15.)

\* To be written out, or repeated to the Teacher.

Let *A* be the given point, and *BC* the given straight line.  
It is required to draw from *A* a straight line equal to *BC*.



## CONSTRUCTION

From the point *A* to *B* draw the straight line *AB*; (post. 1.)  
upon *AB* describe the equilateral triangle *DAB*, (1. 1)  
and produce the straight lines *DA*, *DB*, to *E* and *F*; (post. 11.)  
from the centre *B*, at the distance *BC*, describe the circle *CGH*,  
(post. 3)  
and from the centre *D*, at the distance *DG*, describe the circle *GKL*.

**Then *AL* shall be equal to *BC*.**

## DEMONSTRATION

Because the point *B* is the centre of the circle *CGH*,  
therefore *BC* is equal to *BG*; (def. 15)  
and because *D* is the centre of the circle *GKL*,  
therefore *DL* is equal to *DG*,  
and *DA*, *DB*, parts of them, are equal;  
therefore the remainder *AL* is equal to the remainder *BG*. (ax. 3.)  
But it has been shown that *BC* is equal to *BG*;  
wherefore *AL* and *BC* are each of them equal to *BG*;  
and things which are equal to the same thing are equal to one another;  
therefore the straight line *AL* is equal to *BC*. (ax. 1.)  
Wherefore from the given point *A* a straight line *AL* has been drawn  
equal to the given straight line *BC*.

Which was to be done.

## PROP. III.—PROBLEM.

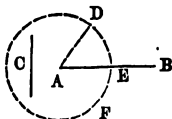
*From the greater of two given straight lines to cut off a part equal to the less.*

(References—Prop. I. 2; def. 15; post. 3; ax. 1.)

Let *AB* and *C* be the given straight lines, of which *AB* is greater than *C*.



It is required to cut off from  $AB$  the greater, a part equal to  $C$ , the less.



#### CONSTRUCTION

From the point  $A$  draw the straight line  $AD$  equal to  $C$ ; (I. 2)  
and from the centre  $A$ , at the distance  $AD$ , describe the circle  $DEF$ .  
(post. 3.)

**Then  $AB$  shall be equal to  $C$ .**

#### DEMONSTRATION

Because  $A$  is the centre of the circle  $DEF$ ,

*therefore  $AE$  is equal to  $AD$ ; (def. 15)*

but the straight line  $C$  is likewise equal to  $AD$ ; (constr.)

*therefore  $AE$  and  $C$  are each of them equal to  $AD$ ;*

**wherefore the straight line  $AE$  is equal to  $C$ . (ax. 1.)**

And therefore from  $AB$ , the greater of two straight lines, a part  $AE$   
has been cut off equal to  $C$ , the less.

Which was to be done.

#### PROP. IV.—THEOREM.

*If two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by these sides equal to each other:*

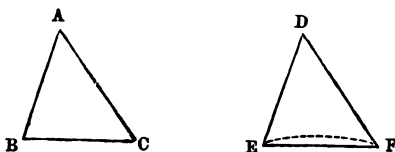
*then they shall likewise have their bases, or third sides, equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.*

(References—Ax. 8, 10.)

Let  $ABC$ ,  $DEF$ , be two triangles, which have the two sides  $AB$ ,  $AC$ , equal to the two sides,  $DE$ ,  $DF$ , each to each, viz.  $AB$  to  $DE$ , and  $AC$  to  $DF$ ;

and the angle  $BAC$  equal to the angle  $EDF$ .

Then the base **BC** shall be equal to the base **EF**;  
 and the triangle **ABC** to the triangle **DEF**,  
 and the other angles to which the equal sides are opposite, shall be equal, each to each, viz. the angle **ABC** to the angle **DEF**, and the angle **ACB** to **DFE**.



## DEMONSTRATION

For, if the triangle **ABC** be applied to the triangle **DEF**,  
 so that the point **A** may be on **D**, and the straight line **AB** upon **DE**;  
 because **AB** is equal to **DE**,

*therefore the point B shall coincide with the point E*;

and **AB** coinciding with **DE**,

and the angle **BAC** being equal to the angle **EDF**, (hyp.)

*therefore AC shall coincide with DF*;

and because **AC** is equal to **DF**,

*wherefore also the point C shall coincide with the point F.*

But the point **B** coincides with the point **E**;

*wherefore the base BC shall coincide with the base EF*;

because the point **B** coinciding with **E**, and **C** with **F**,

if the base **BC** do *not* coincide with the base **EF**,

two straight lines would inclose a space, which is impossible. (ax. 10.)

**Therefore the base BC shall coincide with the base EF,**  
**and be equal to it.**

**Wherefore the whole triangle ABC shall coincide with the**  
**whole triangle DEF, and be equal to it;**

and the other angles of the one shall coincide with the remaining  
 angles of the other, and be equal to them, viz.:

**the angle ABC to the angle DEF,**

**and the angle ACB to the angle DFE.**

Therefore, if two triangles have two sides of the one equal to two sides  
 of the other, &c.

Which was to be demonstrated.

## PROP. V.—THEOREM.

*The angles at the base of an isosceles triangle are equal to one another;*

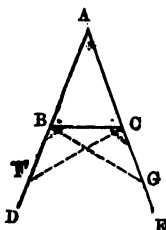
*and if the equal sides be produced, the angles upon the other side of the base shall be equal.*

(References — Prop. I. 3, 4; ax. 3.)

Let  $ABC$  be an isosceles triangle, of which the side  $AB$  is equal to  $AC$ ,

and let the straight lines  $AB$ ,  $AC$ , be produced to  $D$  and  $E$ .

**Then the angle  $ABC$  shall be equal to the angle  $ACB$ ,  
and the angle  $CBD$  to the angle  $BCE$ .**



## CONSTRUCTION

In  $BD$  take any point  $F$ ,

and from  $AE$ , the greater, cut off  $AG$  equal to  $AF$ , the less, (I. 3)  
and join  $FC$ ,  $GB$ .

## DEMONSTRATION

Because  $AF$  is equal to  $AG$ ; (constr.)

and  $AB$  to  $AC$ ; (hypoth.)

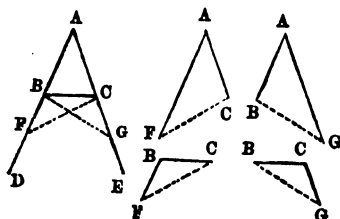
the two sides  $FA$ ,  $AC$ , are equal to the two  $GA$ ,  $AB$ , each to each,

and they contain the angle  $FAG$  common to the two triangles  $AFC$ ,

$AGB$ ;

---

To assist the learner, the figure may be drawn so as to exhibit to the eye the triangles of which it is composed, in the following manner:—



*therefore the base FC is equal to the base GB,  
and the triangle AFC to the triangle AGB;  
and the remaining angles of the one are equal to the remaining  
angles of the other, each to each, to which the equal sides are  
opposite, viz.  
the angle ACF to the angle ABG,  
and the angle AFC to the angle AGB. (I. 4.)*

And because the whole AF, is equal to the whole AG,  
of which the parts AB, AC, are equal ;  
*the remainder BF shall be equal to the remainder CG ; (ax. 3.)*  
and FC was proved to be equal to GB,  
therefore the two sides BF, FC, are equal to the two CG, GB, each to  
each ;  
and the angle BFC is equal to the angle CGB,  
and the base BC is common to the two triangles BFC, CGB ;  
wherefore these triangles are equal, and their remaining angles,  
each to each, to which the equal sides are opposite ;  
*therefore the angle FBC is equal to the angle GCB ;  
and the angle BCF is equal to the angle CBG. (I. 4.)*

And, since it has been demonstrated,  
that the whole angle ABG is equal to the whole angle ACF,  
the parts of which, the angles CBG, BCF, are equal ;  
*therefore the remaining angle ABC is equal to the  
remaining angle ACB,  
which are the angles at the base of the triangle ABC.*

And it has been proved,  
*that the angle FBC is equal to the angle GCB,  
which are the angles upon the other side of the base.*  
Therefore the angles at the base, &c. Q. E. D.

Cor. Hence every equilateral triangle is also equiangular.

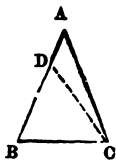
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#### PROP. VI.—THEOREM.

*If two angles of a triangle be equal to each other ;  
then the sides also which subtend, or are opposite to, the equal angles  
shall be equal to one another.*

(References — Prop. I. 3, 4.)

Let the triangle  $ABC$  have the angle  $ABC$  equal to the angle  $ACB$ .  
**Then the side  $AB$  shall be equal to the side  $AC$ .**



## CONSTRUCTION

For, if  $AB$  be not equal to  $AC$ ,  
 one of them is greater than the other.  
 Let  $AB$  be the greater;  
 and from it cut off  $DB$  equal to  $AC$ , the less, (I. 3.) and join  $DC$ .

## DEMONSTRATION

Then in the triangles  $DBC$ ,  $ACB$ ,  
 because  $DB$  is assumed to be equal to  $AC$ , and  $BC$  is common to both,  
 the two sides  $DB$ ,  $BC$ , must be equal to the two  $AC$ ,  $CB$ , each to each;  
 and the angle  $DBC$  is equal to the angle  $ACB$ ; (hyp.)

*therefore the base  $DC$  must be equal to the base  $AB$ ,  
 and the triangle  $DBC$  to the triangle  $ACB$ , (I. 4)  
 the less equal to the greater, which is absurd.*

*Therefore  $AB$  is not unequal to  $AC$ ,*

**that is  $AB$  is equal to  $AC$ .**

Wherefore, if two angles, &c.

Q. E. D.

Cor. Hence every equiangular triangle is also equilateral.

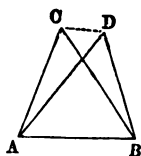
## PROP. VII.—THEOREM.

*Upon the same base and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to each other, and likewise those which are terminated in the other extremity.*

(References — Prop. I. 5 ; ax. 9.)

If it be possible, let there be two triangles,  $ACB$ ,  $ADB$ , upon the same base  $AB$ , and upon the same side of it, which have their sides  $CA$ ,  $DA$ , terminated in the extremity  $A$  of the base, equal to one another, and likewise their sides  $CB$ ,  $DB$ , that are terminated in  $B$ .

*First.* Let the vertex of each of the triangles be without the other.



CONSTRUCTION

Join the vertices C and D by the straight line CD.

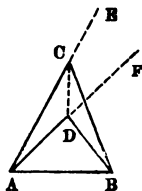
DEMONSTRATION

Because, in the triangle ACD, AC is assumed to be equal to AD,  
*therefore the angle ACD must be equal to the angle ADC ; (I. 5)*  
 but the angle ACD is greater than the angle BCD ; (ax. 9)  
 therefore the angle ADC must be greater also than BCD ;  
*much more then must the angle BDC be greater than the angle BCD.*

Again, because in the triangle BCD, CB is assumed to be equal to BD,

*therefore the angle BDC must be equal to the angle BCD ; (I. 5)*  
 but the angle BDC has been proved to be greater than the angle BCD ;  
*therefore the angle BDC must be both equal to, and greater than the*  
*angle BCD ;*  
 which is impossible.

*Secondly.* Let the vertex D of the triangle ADB fall within the triangle ACB.



CONSTRUCTION

Join the vertices C and D, and produce AC, AD to E, F.

DEMONSTRATION

Because in the triangle ACD, AC is assumed to be equal to AD,  
*therefore the angles ECD, FDC upon the other side of the base CD,*  
*must be equal to one another ; (I. 5)*

but the angle  $ECD$  is greater than the angle  $BCD$ ; (ax. 9)  
 wherefore the angle  $FDC$  must likewise be greater than the angle  $BCD$ ;  
*much more then must the angle  $BDC$  be greater than the angle  $BCD$ .*

Again, because in the triangle  $BCD$ ,  $CB$  is assumed to be equal to  $DB$ ,

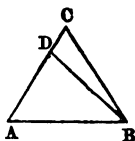
*therefore the angle  $BDC$  must be equal to the angle  $BCD$  ; (I. 5)*

but  $BDC$  has been proved to be greater than  $BCD$  ;

*wherefore the angle  $BDC$  must be both equal to, and greater than, the angle  $BCD$  ;*

which is impossible.

*Thirdly.* Let the vertex of one triangle be upon a side of the other.



This case needs no demonstration, for it is evident from inspection that the side  $AC$  is greater than its part  $AD$ .

Therefore upon the same base and on the same side of it, &c.

Q. E. D.

### PROP. VIII.—THEOREM.

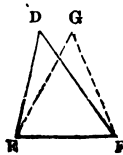
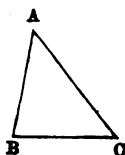
*If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal;*

*then the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.*

(References—Prop. I. 7 ; ax. 8.)

Let  $ABC$ ,  $DEF$  be two triangles, having the two sides  $AB$ ,  $AC$ , equal to the two sides  $DE$ ,  $DF$ , each to each, viz.,  $AB$  to  $DE$ , and  $AC$  to  $DF$  ; and also the base  $BC$  equal to the base  $EF$ .

**Then the angle  $BAC$  shall be equal to the angle  $EDF$ .**



## DEMONSTRATION

For, if the triangle ABC be applied to DEF,  
so that the point B be on E, and the straight line BC upon EF;  
then because BC is equal to EF, (hyp.)

*therefore the point C shall coincide with the point F;*  
wherefore BC coinciding with EF,

BA and AC shall coincide with ED and DF;  
for if the base BC coincides with the base EF, but the sides BA, CA  
do not coincide with the sides ED, FD, but have a different situation as  
EG, FG,

then, upon the same base EF, and upon the same side of it, there can  
be two triangles that have their sides which are terminated in one  
extremity of the base equal to one another, and likewise their sides  
which are terminated in the other extremity;  
but this is impossible. (I. 7.)

Therefore, if the base BC coincides with the base EF,  
*the sides BA, AC, cannot but coincide with the sides ED, DF;*

**wherefore the angle BAC coincides with the angle EDF,  
and is equal to it. (ax. 8.)**

Therefore, if two triangles, &c.

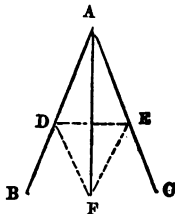
Q. E. D.

## PROP. IX.—PROBLEM.

*To bisect a given rectilineal angle, that is, to divide it into two equal angles.*

(References—Prop. I. 1, 3, 8.)

Let the angle BAC be the given rectilineal angle.  
It is required to bisect it.



## CONSTRUCTION

Take any point D in AB,  
from AC cut off AE equal to AD, (I. 3) and join DE;



upon the side of it opposite to A, describe the equilateral triangle DEF, (I. 1) and join AF.

**Then the straight line AF bisects the angle BAC.**

#### DEMONSTRATION

Because AD is equal to AE, (constr.)  
and AF is common to the two triangles DAF, EAF;  
the two sides DA, AF, are equal to the two sides EA, AF, each to each;  
and the base DF is equal to the base EF; (constr.)

*therefore the angle DAF is equal to the angle EAF.* (I. 8.)

**Wherefore the given rectilineal angle BAC is bisected by the line AF.** Q. E. F.

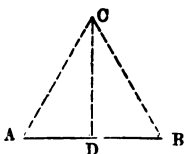
#### PROP. X.—PROBLEM.

*To bisect a given finite straight line, that is, to divide it into two equal parts.*

(References—Prop. I. 1, 4, 9.)

Let AB be the given straight line.

It is required to divide AB into two equal parts.



#### CONSTRUCTION

Upon the straight line AB describe the equilateral triangle ABC, (I. 1) and bisect the angle ACB by the straight line CD. (I. 9.)

**Then AB shall be divided into two equal parts in the point D.**

#### DEMONSTRATION

Because AC is equal to CB, (constr.)  
and CD common to the two triangles ACD, BCD,  
the two sides AC, CD, are equal to the two sides BC, CD, each to each;  
and the angle ACD is equal to the angle BCD; (constr.)

*therefore the base AD is equal to the base DB.* (I. 4.)

**Wherefore the straight line AB is divided into two equal parts in the point D.** Q. E. F.

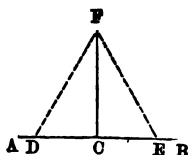
## PROP. XI. — PROBLEM.

*To draw a straight line at right angles to a given straight line, from a given point in the same.*

(References — Prop. I. 1, 3, 8; ax. 1; def. 10.)

Let AB be the given straight line, and C the given point in it.

It is required to draw a straight line from the point C at right angles to AB.



## CONSTRUCTION

In AC, take any point D,  
and make CE equal to CD; (I. 3)  
upon DE describe the equilateral triangle DFE, (I. 1)  
and join FC.

**Then FC, drawn from the point C, shall be at right angles to AB.**

## DEMONSTRATION

Because DC is equal to CE,  
and CF is common to the two triangles DCF, ECF;  
the two sides DC, CF, are equal to the two EC, CF, each to each;  
and the base DF is equal to the base EF; (constr.)

*therefore the angle DCF is equal to the angle ECF; (I. 8)*

and they are adjacent angles.

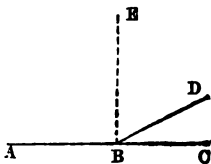
But, 'when a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle;' (def. 10)

*therefore each of the angles DCF, ECF, is a right angle.*

**Wherefore, from the point C, in the straight line AB, FC has been drawn at right angles to AB.**

Q. E. F.

Cor. By help of this problem, it may be demonstrated that two straight lines cannot have a common segment.



If it be possible, let the two straight lines ABC, ABD, have the segment AB common to both of them.

#### CONSTRUCTION

From the point B, draw BE at right angles to AB. (I. 11.)

#### DEMONSTRATION

Because ABC is a straight line,

*therefore the angle CBE is equal to the angle EBA ; (def. 10)*

in the same manner, because ABD is assumed to be a straight line,

*therefore the angle DBE must be equal to the angle EBA ;*

*wherefore the angle DBE must be equal to the angle CBE, (ax. 1)*

the less to the greater, which is impossible.

*Therefore two straight lines cannot have a common segment.*

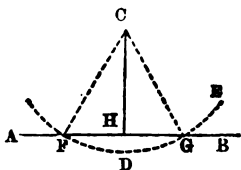
#### PROP. XII.—PROBLEM.

*To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.*

(References — Prop. I. 8, 10 ; post. 3 ; def. 10, 15.)

Let AB be the given straight line, which may be produced to any length both ways, and C the given point without it.

It is required to draw from C a straight line perpendicular to AB.



## CONSTRUCTION

Take any point D, upon the other side of AB,  
and from the centre C, at the distance CD, describe the circle EGF,  
meeting AB in F, G; (post. 3)  
bisect FG in H, (I. 10) and join CF, CH, CG.

**Then the straight line CH, drawn from the point C, is perpendicular to the given straight line AB.**

## DEMONSTRATION

Because FH is equal to HG, (constr.)  
and HC common to the two triangles FHC, GHC,  
the two sides FH, HC, are equal to the two GH, HC, each to each;  
and the base CF is equal to the base CG; (def. 15)

*therefore the angle CHF is equal to the angle CHG; (I. 8)*

and they are adjacent angles.

But 'when a straight line standing on a straight line makes the adjacent angles equal to one another, each of them is a right angle; and the straight line which stands upon the other is called a perpendicular to it.' (def. 10.)

**Therefore CH is perpendicular to AB,**

and it has been drawn from the point C.

Q. E. F.

## PROP. XIII.—THEOREM.

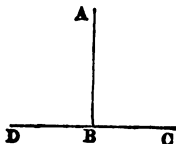
*The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.*

(References—Prop. I. 11; ax. 1, 2; def. 10.)

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD.

**Then the angles CBA, ABD are either two right angles, or are together equal to two right angles.**

*First, let them be equal to one another.*

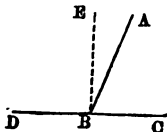


## DEMONSTRATION

Then, since the straight line AB, standing on the straight line CD, makes the adjacent angles ABC, ABD equal to one another, *therefore each of them is a right angle*; (def. 10)

**wherefore the angles ABC, ABD, are two right angles.**

*Secondly*, let them not be equal to one another.



## CONSTRUCTION

From the point B draw BE at right angles to CD. (I. 11.)

## DEMONSTRATION

Because BE is at right angles to CD, (constr.)

*therefore the angles CBE, EBD are two right angles*; (def. 10)

and because the angle CBE is equal to the two angles CBA, ABE, add to each of these equals the angle EBD;

*then the angles CBE, EBD, are equal to the three angles CBA, ABE, EBD.* (ax. 2.)

Again, because the angle DBA is equal to the two angles DBE, EBA, add to each of these equals the angle ABC;

*then the angles DBA, ABC, are equal to the angles DBE, EBA, ABC.* (ax. 2.)

But the angles CBE, EBD, have been proved to be equal to the same three angles;

and things which are equal to the same thing are equal to one another,

*therefore the angles CBE, EBD, are equal to the angles DBA, ABC;* (ax. 1)

but the angles CBE, EBD are two right angles; (constr.)

**therefore the angles DBA, ABC, are together equal to two right angles.** (ax. 1.)

Wherefore when a straight line, &c.

Q. E. D.

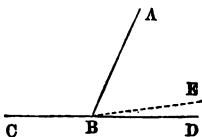
## PROP. XIV.—THEOREM.

*If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines shall be in one and the same straight line.*

(References—Prop. I. 13; ax. 1, 3.)

At the point B in the straight line AB, let BC, BD, upon the opposite sides of AB, make the adjacent angles ABC, ABD, equal to two right angles.

**Then CB shall be in the same straight line with ED.**



## CONSTRUCTION

For if BD be not in the same right line with BC,  
let BE be in the same right line with it.

## DEMONSTRATION

Because CBE is assumed to be a straight line, and AB meets it in B,  
*therefore the adjacent angles ABC, ABE, must be together equal to two right angles; (I. 13)*

but the angles ABC, ABD, are together equal to two right angles;  
(hyp.)

*therefore the angles CBA, ABE, must be equal to the angles CBA, ABD; (ax. 1)*

take away the common angle ABC,

*therefore the remaining angle ABE must be equal to the remaining angle ABD; (ax. 3)*

the less to the greater, which is impossible;

*therefore BE is not in the same straight line with BC.*

And, in like manner, it may be proved, that no other can be in the same straight line with it but BD,

**therefore ED is in the same straight line with EC.**

Wherefore, if at a point, &c.

Q. E. D.

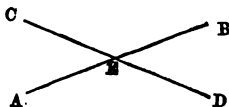
## PROP. XV.—THEOREM.

*If two straight lines cut each other, then the vertical or opposite angles shall be equal.*

(References—Prop. I. 13; ax. 1, 3.)

Let the two straight lines AB, CD, cut each other in the point E.

**Then the angle AEC shall be equal to the angle DEB, and the angle CEB to the angle AED.**



## DEMONSTRATION

Because the straight line AE makes with CD the angles CEA, AED, *these angles are together equal to two right angles.* (I. 13.)

Again, because the straight line DE makes with AB the angles AED, DEB,

*these angles are also together equal to two right angles;*

and CEA, AED, have been proved to be together equal to two right angles;

*wherefore the angles CEA, AED, are equal to the angles AED, DEB;*  
(ax. 1)

take away the common angle AED,

**then the remaining angle CEA is equal to the remaining angle DEB.** (ax. 3.)

In the same manner it can be shown,

**that the angle CEB is equal to the angle AED.**

Therefore, if two straight lines, &c.

Q. E. D.

Cor. 1. From this it is manifest, that, if two straight lines cut one another, the angles they make at the point where they cut, are together equal to four right angles.

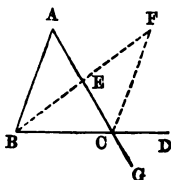
Cor. 2. And, consequently, that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

## PROP. XVI.—THEOREM.

*If one side of a triangle be produced, then the exterior angle is greater than either of the interior opposite angles.*

(References—Prop. I. 3, 4, 10, 15; ax. 9.)

Let  $ABC$  be a triangle, and let the side  $BC$  be produced to  $D$ .  
**Then the exterior angle  $ACD$  shall be greater than either of the angles  $ABC$  or  $CAB$ .**



## CONSTRUCTION

Bisect  $AC$  in  $E$ , (I. 10) and join  $BE$ ,  
 produce  $BE$  to  $F$ , and make  $EF$  equal to  $BE$ ; (I. 3)  
 and join  $FC$ .

## DEMONSTRATION

Because  $AE$  is equal to  $EC$ , and  $BE$  to  $EF$ ;  
 the two sides  $AE$ ,  $EB$ , are equal to the two  $CE$ ,  $EF$ , each to each;  
 and the angle  $AEB$  is equal to the angle  $CEF$ ,  
 because they are opposite, vertical angles; (I. 15)

*therefore the base  $AB$  is equal to the base  $CF$ ; (I. 4)*

*and the triangle  $AEB$  to the triangle  $CEF$ ,*

*and the remaining angles to the remaining angles, each to each, to  
 which the equal sides are opposite;*

*wherefore the angle  $BAE$  is equal to the angle  $ECF$ ;*

but the angle  $ECD$  is greater than the angle  $ECF$ ;

**therefore the angle  $ACD$  is greater than the angle  $BAC$ .**

In the same manner, if the side  $BC$  be bisected, and  $AC$  be produced  
 to  $G$ , it may be demonstrated that the angle  $BCG$ ,

**that is, the angle  $ACD$ , (I. 15) is greater than the angle  $ABC$ .**

Therefore, if one side, &c.

Q. E. D.

## PROP. XVII.—THEOREM.

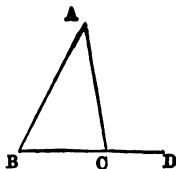
*Any two angles of a triangle are together less than two right angles.*

(References—Prop. I. 13, 16; ax. 4.)

Let  $ABC$  be any triangle.



**Then any two of its angles shall be together less than two right angles.**



**CONSTRUCTION**

Produce the side BC to D.

**DEMONSTRATION**

Because ACD is the exterior angle of the triangle ABC,  
*therefore the angle ACD is greater than the interior and opposite angle ABC; (I. 16)*

to each of these unequals add the angle ACB;

*then the angles ACD, ACB, are greater than the angles ABC, ACB;*

but the angles ACD, ACB, are together equal to two right angles;  
 (I. 13)

**therefore the angles ABC, BCA, are less than two right angles.**

In like manner, it may be proved,

**that the angles BAC, ACB, are less than two right angles,**

**and likewise the angles CAB, ABC.**

Therefore any two angles, &c.

Q. E. D.

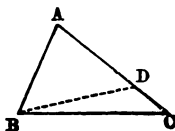
**PROP. XVIII.—THEOREM.**

*The greater side of every triangle is opposite to the greater angle; i.e., in any triangle, if one side be greater than another, then the angle which is opposite the greater side is greater than the angle which is opposite the less.*

(References — Prop. I. 3, 5, 16.)

Let ABC be any triangle of which the side AC is greater than the side AB.

**Then the angle ABC shall be greater than the angle BCA.**



## CONSTRUCTION

Because the side AC is greater than the side AB,  
make AD equal to AB, (I. 3.) and join BD.

## DEMONSTRATION

Because the angle ADB is the exterior angle of the triangle BDC,  
*therefore the angle ADB is greater than the interior and opposite angle*  
DCB; (I. 16)

but the angle ADB is equal to the angle ABD, (I. 5)  
because the side AB is equal to the side AD,

*therefore the angle ABD is likewise greater than the angle ACB;*

**wherefore much more is the angle ABC greater than the angle ACB.**

Therefore the greater side, &c.

Q. E. D.

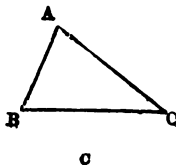
## PROP. XIX.—THEOREM.

*The greater angle of every triangle is subtended by the greater side, i.e., in any triangle, if one angle be greater than another, then the side which is opposite the greater angle is greater than the side which is opposite to the less.*

(References—Prop. I. 5, 18.)

Let ABC be any triangle, of which the angle ABC is greater than the angle BCA.

**Then the side AC shall be greater than the side AB.**



For, if  $AC$  be not greater than  $AB$ ,  
 $AC$  must either be equal to  $AB$ , or less than it.  
 But  $AC$  is not equal to  $AB$ ,  
 because then the angle  $ABC$  would be equal to the angle  $ACB$ ; (I. 5)  
 but it is not, (hyp.)  
*therefore  $AC$  is not equal to  $AB$ .*

Neither is  $AC$  less than  $AB$ ;  
 because then the angle  $ABC$  would be less than the angle  $ACB$ ; (I. 18)  
 but it is not; (hyp.)  
*therefore  $AC$  is not less than  $AB$ ;*  
 and it has been shown that it is not equal to  $AB$ ;  
**therefore  $AC$  is greater than  $AB$ .**  
 Wherefore, in any triangle, &c. Q. E. D.

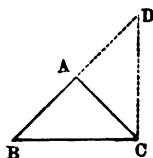
### PROP. XX.—THEOREM.

*Any two sides of a triangle are together greater than the third side.*

(References—Prop. I. 3, 5, 19; ax. 9.)

Let  $ABC$  be a triangle.

Then any two sides of it together are greater than the third side, viz.,  
 the sides  $BA$ ,  $AC$ , greater than the side  $BC$ ;  
 $AB$ ,  $BC$ , greater than  $AC$ ;  
 and  $BC$ ,  $CA$ , greater than  $AB$ .



#### CONSTRUCTION

Produce  $BA$  to the point  $D$ ,  
 and make  $AD$  equal to  $AC$ ; (I. 3) and join  $DC$ .

#### DEMONSTRATION

Because  $DA$  is equal to  $AC$ ,  
*therefore the angle  $ADC$  is equal to the angle  $ACD$ ; (I. 5)*  
 but the angle  $BCD$  is greater than the angle  $ACD$ ;

*therefore the angle BCD is greater than the angle ADC ;*  
 and because in the triangle DCB, the angle BCD is greater than the angle BDC,  
 and that the side which is opposite the greater angle is greater than that which is opposite the less ; (I. 19)

*therefore the side DB is greater than the side BC ;*  
 but DB is equal to BA and AC ; (constr.)

**therefore the sides BA, AC, are greater than BC.**

In the same manner, it may be proved,

**that the sides AB, BC, are greater than CA ;**

**and BC, CA, are greater than AB.**

Therefore, any two sides, &c.

Q. E. D.

# PROP. XXI.—THEOREM.

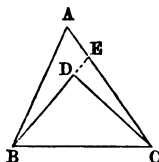
*If from the ends of a side of a triangle there be drawn two straight lines to a point within the triangle ;*  
*then these shall be less than the other two sides of the triangle, but shall contain a greater angle.*

(References — I. 16, 20 ; ax. 4.)

Let the two straight lines BD, CD, be drawn from B, C, the ends of the side BC of the triangle ABC, to the point D within it.

**Then BD and DC shall be less than the two sides BA, AC, of the triangle,**

**but shall contain an angle BDC greater than the angle BAC.**



## CONSTRUCTION

Produce BD to meet AC in E.

## DEMONSTRATION

Because two sides of a triangle are greater than the third side, (I. 20)

*therefore the two sides BA, AE, of the triangle ABE, are greater than BE ;*

to each of these add EC ;

*then the sides BA, AC, are greater than BE, EC. (ax. 4.)*

Again, because the two sides CE, ED, of the triangle CED, are greater than CD,

add DB to each of these ;

*then the sides CE, EB, are greater than CD, DB ; (ax. 4)*

but it has been shown that BA, AC, are greater than BE, EC,

**much more then are BA, AC, greater than BD, DC.**

Again, because the exterior angle of a triangle is greater than the interior and opposite angle ; (I. 16)

*therefore the exterior angle BDC of the triangle CDE is greater than CED ;*

for the same reason,

*the exterior angle CEB of the triangle ABE is greater than BAC ;*

and it has been shown that the angle BDC is greater than CEB ;

**much more then is the angle BDC greater than the angle BAC.**

Therefore, if from the ends, &c.

Q. E. D.

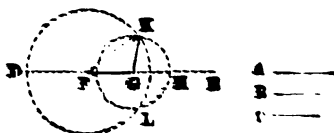
### PROP. XXII.—PROBLEM.

*To make a triangle of which the sides shall be equal to three given straight lines, but of which any two whatever must be greater than the third.*

(References—Prop. I. 3 ; post. 3 ; ax. 1 ; def. 15.)

Let A, B, C be three given right lines, of which any two whatever are greater than the third — viz., A and B together greater than C ; A and C together greater than B ; and B and C greater than A.

It is required to make a triangle having its sides equal to A, B, C, each to each.



CONSTRUCTION

Take a straight line DE terminated at the point D, but unlimited towards E,

make DF equal to A, FG equal to B, and GH equal to C; (I. 3)  
 and from the centre F, at the distance FD, describe the circle DKL;  
 (post. 3)  
 and from the centre G, at the distance GH, describe the circle HLK;  
 and join KF, KG.

**Then the triangle KFG shall have its sides equal to the three straight lines A, B, C.**

DEMONSTRATION

Because the point F is the centre of the circle DKL,

*therefore FD is equal to FK; (def. 15)*

but FD is equal to the straight line A; (constr.)

*therefore FK is equal to A. (ax. 1)*

Again, because G is the centre of the circle LKH,

*therefore GH is equal to GK; (def. 15)*

but GH is equal to C; (constr.)

*therefore GK is equal to C;*

and FG is equal to B; (constr.)

*therefore the three straight lines KF, FG, GK, are equal to the three A, B, C.*

**And therefore the triangle KFG has its sides KF, FG, GK, equal to the given straight lines A, B, C. Q. E. F.**

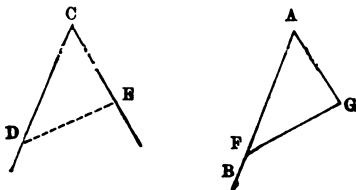
PROP. XXIII.—PROBLEM.

*At a given point in a given straight line to construct a rectilineal angle equal to a given rectilineal angle.*

(References—Prop. I. 8, 22.)

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle.

It is required to make an angle at the point A in the straight line AB, equal to the rectilineal angle DCE.



## CONSTRUCTION

Take in  $CD$ ,  $CE$ , any points  $D$ ,  $E$ , and join  $DE$ ;  
make the triangle  $AFG$ , the sides of which shall be equal to the three straight lines  $CD$ ,  $DE$ ,  $CE$ , so that  $CD$  be equal to  $AF$ ,  $CE$  to  $AG$ , and  $DE$  to  $FG$ . (I. 22.)

**Then the angle  $FAG$  shall be equal to the angle  $DCE$ .**

## DEMONSTRATION

Because  $DC$ ,  $CE$ , are equal to  $FA$ ,  $AG$ , each to each,  
and the base  $DE$  to the base  $FG$ ; (constr.)

**therefore the angle  $DCE$  is equal to the angle  $FAG$ .** (I. 8.)

Wherefore, at the given point  $A$  in the straight line  $AB$ , the angle  $FAG$  is made equal to the given rectilineal angle  $DCE$ . Q. E. F.

## PROP. XXIV.—THEOREM.

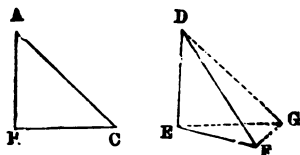
*If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides, equal to them, of the other;*

*then the base of that which has the greater angle, shall be greater than the base of the other.*

(References—Prop. I. 3, 4, 5, 19, 23; ax. 9.)

Let  $ABC$ ,  $DEF$ , be two triangles which have the two sides  $AB$ ,  $AC$ , equal to the two  $DE$ ,  $DF$ , each to each—viz.,  $AB$  equal to  $DE$ , and  $AC$  to  $DF$ ; but the angle  $BAC$  greater than the angle  $EDF$ .

**Then the base  $BC$  shall be greater than the base  $EF$ .**



## CONSTRUCTION

(Of the two sides  $DE$ ,  $DF$ , let  $DE$  be the side which is not greater than the other,

and at the point D, in the straight line DE, make the angle EDG equal to the angle BAC; (I. 23)  
and make DG equal to AC or DF, (I. 3) and join EG, GF.

## DEMONSTRATION

Because AB is equal to DE, and AC to DG,  
the two sides BA, AC, are equal to the two ED, DG, each to each,  
and the angle BAC is equal to the angle EDG; (constr.)  
*therefore the base BC is equal to the base EG.* (I. 4.)

And, because DG is equal to DF,  
*therefore the angle DFG is equal to the angle DGF;* (I. 5)  
but the angle DGF is greater than the angle EGF,  
therefore the angle DFG is greater than the angle EGF;  
*much more then is the angle EFG greater than the angle EGF.*

And, because the angle EFG of the triangle EFG, is greater than its angle EGF,  
and that the greater side is opposite to the greater angle; (I. 19)  
*therefore the side EG is greater than the side EF;*  
but EG has been proved to be equal to BC;  
**therefore BC is greater than EF.**

Therefore, if two triangles, &c.

Q. E. D.

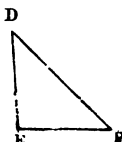
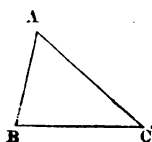
## PROP. XXV.—THEOREM.

*If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other;  
then the angle contained by the sides of the one which has the greater base, shall be greater than the angle contained by the sides, equal to them, of the other.*

(References—Prop. I. 4, 24.)

Let ABC, DEF, be two triangles which have the two sides AB, AC, equal to the two sides DE, DF, each to each—viz., AB equal to DE, and AC to DF; but the base CB greater than the base EF.

**Then the angle BAC shall be greater than the angle EDF.**





## DEMONSTRATION

For if the angle BAC be not greater than the angle EDF, it must either be equal to it or less.

Now, if the angle BAC be equal to the angle EDF,

*then must the base BC be equal to the base EF; (I. 4)*

but it is not; (hyp.)

*therefore the angle BAC is not equal to the angle EDF.*

Again, if the angle BAC be less than the angle EDF,

*then must the base BC be less than the base EF; (I. 24)*

but it is not; (hyp.)

*therefore the angle BAC is not less than the angle EDF;*

and it has been proved it is not equal to it;

**therefore the angle BAC is greater than the angle EDF.**

Wherefore, if two triangles, &c.

Q. E. D.

## PROP. XXVI.—THEOREM.

*If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side; viz., either the side adjacent to the equal angles in each, or the side opposite to them;*

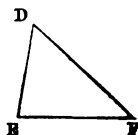
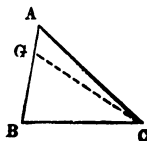
*then the other sides shall be equal, each to each, and also the third angle of the one equal to the third angle of the other.*

(References — Prop. I. 3, 4, 16; ax. 1.)

Let the two triangles ABC, DEF, have the angles ABC, BCA, equal to the angles DEF, EFD, each to each; also one side equal to one side.

*First.* Let the sides adjacent to the equal angles in each be equal; viz., BC equal to EF.

**Then shall the side AB be equal to the side DE, the side AC to the side DF, and the angle BAC to the angle EDF.**



## CONSTRUCTION

For if AB be not equal to DE,  
 one of them is greater than the other.  
 Let AB be the greater of the two,  
 and make BG equal to DE, (I. 3) and join GC.

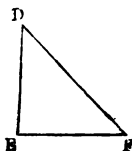
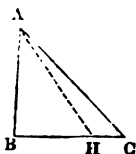
## DEMONSTRATION

Then in the two triangles GBC, DEF,  
 because BG is assumed to be equal to DE,  
 and BC is equal to EF, (hyp.)  
 the two sides GB, BC, must be equal to the two DE, EF, each to each;  
 and the angle GBC is equal to the angle DEF; (hyp.)  
 therefore the base GC must be equal to the base DF,  
 and the triangle GBC to the triangle DEF,  
 and the remaining angles of the one equal to the remaining angles of  
 the other, each to each, to which the equal sides are opposite;  
 therefore the angle GCB must be equal to the angle DFE; (I. 4)  
 but the angle DFE is, by the hypothesis, equal to the angle BCA;  
 wherefore also the angle BCG must be equal to the angle BCA, (ax. 1)  
 the less to the greater, which is impossible;  
 therefore AB is not unequal to DE,  
 that is, AB is equal to DE.

Therefore in the triangles ABC, DEF,  
 because the two AB, BC, are equal to the two DE, EF, each to each;  
 and the angle ABC is equal to the angle DEF; (hyp.)  
 therefore the base AC is equal to the base DF,  
 and the third angle BAC to the third angle EDF. (I. 4.)

Next, let the sides which are opposite to the equal angles in each triangle be equal to one another, viz., AB to DE.

Then likewise in this case the other sides shall be equal,  
 AC to DF, and BC to EF; and also the third angle BAC  
 to the third angle EDF.



## CONSTRUCTION

For if  $BC$  be not equal to  $EF$ ,  
 one of them must be greater than the other.  
 Let  $BC$  be the greater,  
 make  $BH$  equal to  $EF$ , (I. 3) and join  $AH$ .

## DEMONSTRATION

Then, in the two triangles  $ABH$ ,  $DEF$ ,  
 because  $BH$  is assumed to be equal to  $EF$ ,  
 and  $AB$  is equal to  $DE$ , (hyp.)  
 the two sides  $AB$ ,  $BH$ , must be equal to the two  $DE$ ,  $EF$ , each to each;  
 and the angle  $ABH$  is equal to the angle  $DEF$ ; (hyp.)  
 therefore the base  $AH$  must be equal to the base  $DF$ ,  
 and the triangle  $ABH$  to the triangle  $DEF$ ,  
 and the remaining angles of the one equal to the remaining angles of  
 the other, each to each, to which the equal sides are opposite;  
*therefore the angle  $BHA$  is equal to the angle  $EFD$ ; (I. 4)*  
 but the angle  $EFD$  is equal to the angle  $BCA$ ; (hyp.)  
*therefore the angle  $BHA$  must be equal to the angle  $BCA$ ,*  
 that is, the exterior angle,  $BHA$ , of the triangle  $ABC$ , must be equal  
 to its interior and opposite angle  $BCA$ ,  
 which is impossible; (I. 16)  
*wherefore  $BC$  is not unequal to  $EF$ ,*  
**that is,  $BC$  is equal to  $EF$ .**

Therefore in the triangles  $ABC$ ,  $DEF$ ,  
 because  $AB$  is equal to  $DE$ , (hyp.)  
 and  $BC$  has been shown to be equal to  $EF$ ,  
 the two  $AB$ ,  $BC$ , are equal to the two  $DE$ ,  $EF$ , each to each;  
 and the angle  $ABC$  is equal to the angle  $DEF$ ; (hyp.)  
**therefore the base  $AC$  is equal to the base  $DF$ , and**  
**the third angle  $BAC$  to the third angle  $EDF$ . (I. 4.)**  
 Wherefore, if two triangles, &c. Q. E. D.

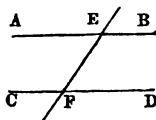
## PROP. XXVII.—THEOREM.

*If a straight line, falling upon two other straight lines, make the  
 alternate angles equal to one another,  
 then these two straight lines shall be parallel.*

(References—Prop. I. 16; def. 35.)

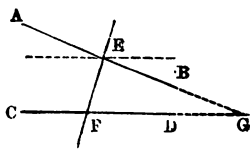
Let the straight line  $EF$ , which falls upon the two straight lines,  $AB$ ,  $CD$ , make the alternate angles,  $AEF$ ,  $EFD$ , equal to one another.

**Then  $AB$  shall be parallel to  $CD$ .**



## CONSTRUCTION

For, if  $AB$  be not parallel to  $CD$ ,  
 $AB$  and  $CD$  being produced, shall meet either towards  $B$ ,  $D$ , or towards  $A$ ,  $C$ ;  
 let them be produced and meet towards  $B$ ,  $D$  in the point  $G$ .



## DEMONSTRATION

Then  $GEF$  must be a triangle,

*and its exterior angle  $AEF$  must be greater than the interior and opposite angle  $EFG$ ; (I. 16)*

but the angle  $AEF$  is equal to the angle  $EFG$ ; (hyp.)

*therefore the angle  $AEF$  must be both greater than, and equal to the angle  $EFG$ ;*

which is impossible.

*Therefore  $AB$ ,  $CD$ , being produced, do not meet towards  $B$ ,  $D$ .*

In like manner it may be demonstrated that

*$AB$ ,  $CD$ , do not meet towards  $A$ ,  $C$ .*

But those straight lines, in the same plane, which meet neither way, though produced ever so far, are parallel to one another; (def. 35)

**therefore  $AB$  is parallel to  $CD$ .**

Wherefore, if a straight line, &c.

**Q. E. D.**

## PROP. XXVIII.—THEOREM.

*If a straight line falling upon two other straight lines, make the exterior angle equal to the interior and opposite, upon the same side of the line, or make the interior angles upon the same side, together equal to two right angles;*

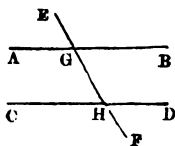
*then the two straight lines shall be parallel to one another.*

(References — Prop. I. 13, 15, 27; ax. I. 3.)

Let the straight line EF, which falls upon the two straight lines AB, CD, make the exterior angle EGB equal to the interior and opposite angle GHD upon the same side;

or make the interior angles on the same side BGH, GHD, together equal to two right angles.

**Then AB shall be parallel to CD.**



## DEMONSTRATION

Because the angle EGB is equal to the angle GHD, (hyp.)

and the angle EGB is equal to the angle AGH, (I. 15)

*therefore the angle AGH is equal to the angle GHD; (ax. 1)*

and they are alternate angles,

**therefore AB is parallel to CD. (I. 27)**

Again, because the angles BGH, GHD, are equal to two right angles; (hyp.)

and that the angles AGH, BGH, are also equal to two right angles; (I. 13)

*therefore the angles AGH, BGH, are equal to the angles BGH, GHD; (ax. 1)*

take away the common angle BGH;

*therefore the remaining angle AGH is equal to the remaining angle GHD; (ax. 3)*

and they are alternate angles;

**therefore AB is parallel to CD. (I. 27)**

*Wherefore, if a straight line, &c.*

Q. E. D.

## PROP. XXIX.—THEOREM.

*If a straight line fall upon two parallel straight lines,  
then it makes the alternate angles equal to each other; and the exterior angle  
equal to the interior and opposite angle upon the same side; and likewise  
the two interior angles upon the same side together equal to two right angles.*

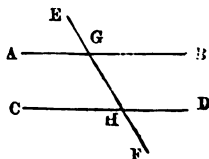
(References — Prop. I. 13, 15; ax. I. 2, 4, 12.)

Let the right line EF, fall on the parallel lines AB, CD.

**Then the alternate angle AGH shall be equal to the  
alternate angle GHD;**

**and the exterior angle EGB shall be equal to the interior  
and opposite angle GHD, on the same side of the line EF;**

**and the two interior angles BGH, GHD, on the same side,  
shall be together equal to two right angles,**



## DEMONSTRATION

For if the angle AGH be not equal to GHD,  
one of them must be greater than the other.

Let AGH be the greater.

Then because the angle AGH is assumed to be greater than the angle  
GHD,

add to each of them the angle BGH;

*therefore the angles AGH, BGH, must be greater than the angles BGH,  
GHD; (ax. 4)*

but the angles AGH, BGH, are equal to two right angles; (I. 13)

*therefore the angles BGH, GHD, must be less than two right angles;*

but 'those straight lines which with another straight line falling upon  
them, make the interior angles on the same side less than two right  
angles, will meet together if continually produced;' (ax. 12)

*therefore the straight lines AB, CD, must meet if produced far enough.*

but they never meet, since they are parallel by the hypothesis;

*therefore the angle AGH is not unequal to the angle GHD,*

**that is, the angle AGH is equal to the angle GHD,  
which are the alternate angles.**

But the angle AGH is equal to the angle EGB; (I. 15)

**therefore the angle EGB is equal to the angle GHD, (ax. 1)**  
**that is, the exterior angle is equal to the interior and**  
**opposite.**

Add to each of these the angles BGH;

*therefore the angles EGB, BGH, are equal to the angles BGH, GHD;*  
 (ax. 2)

but EGB, BGH are equal to two right angles; (I. 13)

**therefore also BGE, GHD, are equal to two right angles,**  
 (ax. 1)

**which are the two interior angles.**

Wherefore, if a straight line, &c.

Q. E. D.

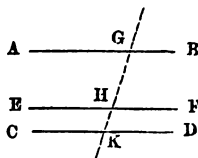
### PROP. XXX.—THEOREM.

*Straight lines which are parallel to the same straight line are parallel to one another.*

(References—Prop. I. 27, 29; ax. 1.)

Let AB, CD, be each of them parallel to EF.

**Then AB shall be also parallel to CD.**



#### CONSTRUCTION

Let the straight line GHK cut AB, EF, CD.

#### DEMONSTRATION

Because GHK cuts the parallel straight lines AB, EF,

*therefore the angle AGH is equal to the angle GHF. (I. 29.)*

Again, because the straight line GHK cuts the parallel straight lines EF, CD,

*therefore the angle GHF is equal to the angle GKD;*

and it was shown that the angle AGH is equal to the angle GHF;

*therefore also the angle AGH is equal to the angle GKD; (ax. 1)*

and they are alternate angles;

**therefore AB is parallel to CD. (I. 27)**

Wherefore, straight lines, &c.

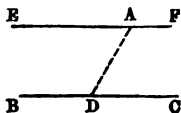
Q. E. D.

## PROP. XXXI.—PROBLEM.

*To draw a straight line through a given point, parallel to a given straight line.*

(References—Prop. I. 23, 27.)

Let A be the given point, and BC the given right line.  
It is required to draw a straight line through the point A, parallel to BC.



## CONSTRUCTION

In BC take any point D, and join AD;  
at the point A, in the straight line AD, make the angle DAE equal to the angle ADC; (I. 23)  
and produce the straight line EA to F.

**Then EF shall be parallel to BC.**

## DEMONSTRATION

Because the straight line AD, which meets the two straight lines BC, EF,

makes the alternate angles EAD, ADC, equal to one another,

**therefore EF is parallel to BC.** (I. 27.)

Wherefore the straight line EAF is drawn through the given point A parallel to the given straight line BC. Q. E. F.

## PROP. XXXII.—THEOREM.

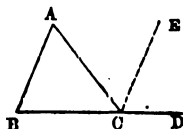
*If the side of any triangle be produced;  
then the exterior angle is equal to the two interior and opposite angles;  
and the three interior angles of every triangle are together equal to two right angles.*

(References—Prop. I. 13, 29, 31; ax. 2.)

Let ABC be a triangle, and let one of its sides BC be produced to D.  
**Then the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC;**



and the three interior angles of the triangle—viz.,  $\angle ABC$ ,  $\angle BCA$ ,  $\angle CAB$ , are together equal to two right angles.



## CONSTRUCTION

Through the point C draw CE parallel to the straight line AB. (I. 31.)

## DEMONSTRATION

Because AB is parallel to CE, and AC meets them,  
therefore the alternate angles  $\angle BAC$ ,  $\angle ACE$ , are equal. (I. 29.)

Again, because AB is parallel to CE, and BD falls upon them,  
therefore the exterior angle  $\angle ECD$  is equal to the interior and opposite angle  $\angle ABC$ ; (I. 29)

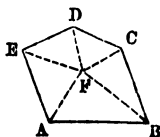
but the angle  $\angle ACE$  was shown to be equal to the angle  $\angle BAC$ ;  
therefore the whole exterior angle  $\angle ACD$  is equal to the two interior and opposite angles  $\angle CAB$ ,  $\angle ABC$ . (ax. 2.)

To each of these equals add the angle  $\angle ACB$ ;  
therefore the angles  $\angle ACD$ ,  $\angle ACB$ , are equal to the three angles  $\angle CBA$ ,  $\angle BAC$ ,  $\angle ACB$ ;

but the angles  $\angle ACD$ ,  $\angle ACB$ , are equal to two right angles; (I. 13)  
therefore also the angles  $\angle CBA$ ,  $\angle BAC$ ,  $\angle ACB$ , are equal to two right angles. (ax. 1.)

Wherefore, if a side of a triangle, &c. Q. E. D.

Cor. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.



## DEMONSTRATION

For, any rectilineal figure  $ABCDE$  can, by drawing straight lines from a point  $F$  within the figure to each angle, be divided into as many triangles as the figure has sides.

By the proposition, the angles of each triangle are equal to two right angles;

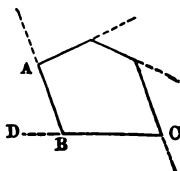
*therefore all the angles of the triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure;*

but the same angles are equal to the angles of the figure, together with the angles at the point F;

and the angles at F, which is the common vertex of all the triangles, are equal to four right angles, (I. 15, cor. 2)

*therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*

**Cor. 2.** *All the exterior angles of any rectilineal figure are together equal to four right angles.*



#### DEMONSTRATION

For, because each interior angle ABC, together with its adjacent exterior angle ABD, are equal to two right angles, (I. 13)

*therefore all the interior angles, together with all the exterior angles, are equal to twice as many right angles as there are angles or sides.*

but all the interior angles, together with four right angles, are equal to twice as many right angles as the figure has sides; (cor. 1)

*therefore all the interior angles, together with all the exterior angles, are equal to all the interior angles and four right angles; (ax. 1)*

take away the interior angles which are common;

*therefore all the exterior angles are equal to four right angles. (ax. 3.)*

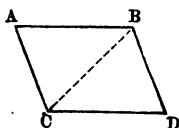
## PROP. XXXIII.—THEOREM.

*The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.*

(References—Prop. I. 4, 27, 29.)

Let AB, CD, be two equal and parallel straight lines, and joined towards the same parts by the straight lines AC, BD.

**Then AC, BD, shall be equal and parallel.**



CONSTRUCTION

Join the opposite points B and C.

## DEMONSTRATION

Because AB is parallel to CD, and BC meets them,  
*therefore the alternate angles ABC, BCD, are equal; (I. 29)*  
 and because AB is equal to CD, (hyp.) and BC common to the two triangles ABC, DCB,  
 the two sides AB, BC, are equal to the two DC, CB;  
 and the angle ABC has been shown to be equal to the angle BCD;  
**therefore the base AC is equal to the base BD, (I. 4)**  
 and the triangle ABC to the triangle BCD,  
 and the other angles to the other angles, each to each, to which the equal sides are opposite;  
*therefore the angle ACB is equal to the angle CBD.*  
 And because the straight line BC meets the two straight lines AC, BD,  
 and makes the alternate angles ACB, CBD, equal to one another,  
**therefore AC is parallel to BD, (I. 27)**  
**and AC was shown to be equal to BD.**  
 Therefore, straight lines, &c. Q. E. D.

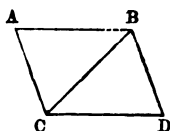
## PROP. XXXIV.—THEOREM.

*The opposite sides and angles of parallelograms are equal to each other, and the diameter bisects them, that is, divides them into two equal parts.*

N.B. A parallelogram is a four-sided figure, of which the opposite sides are parallel; and the diameter is the straight line joining two of its opposite angles.

(References—Prop. I. 4, 26, 29; ax. 2.)

Let ACDB be a parallelogram, of which BC is a diameter.  
**Then the opposite sides and angles of the figure shall be equal to one another;**  
**and the diameter shall bisect it.**



## DEMONSTRATION

Because AB is parallel to CD, and BC meets them,  
*therefore the alternate angles ABC, BCD, are equal to one another;*  
 (I. 29)

and because AC is parallel to BD, and BC meets them,  
*therefore the alternate angles ACB, CBD, are equal to one another.*

Wherefore in the two triangles ABC, CBD,  
 because the two angles ABC, BCA, in the one, are equal to the two  
 angles BCD, CBD, in the other, each to each,  
 and one side BC, which is adjacent to their equal angles, is common  
 to the two triangles;  
 therefore their other sides shall be equal, each to each,  
 and the third angle of the one to the third angle of the other  
 (I. 26)—viz.,

*the side AB to the side CD, and AC to BD,*  
*and the angle BAC to the angle BDC.*

And because the angle ABC is equal to the angle BCD, and the angle  
 CBD to the angle ACB;

*therefore the whole angle ABD is equal to the whole angle ACD;*  
 (ax. 2)

and the angle BAC has been shown to be equal to the angle BDC;

**therefore the opposite sides and angles of a parallelogram  
 are equal to one another.**

**Also the diameter shall bisect it.**

For because AB is equal to CD, and BC common,  
 the two AB, BC, are equal to the two DC, CB, each to each;  
 and the angle ABC has been proved equal to the angle BCD;

*therefore the triangle ABC is equal to the triangle BCD; (I. 4)*

**and the diameter BC divides the parallelogram ACDB  
 into two equal parts.**

Q. E. D.

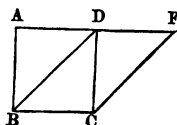
## PROP. XXXV.—THEOREM.

*Parallelograms upon the same base, and between the same parallels, are equal to each other.*

(References—Prop. I. 4, 29, 34; ax. 1, 3, 6.)

Let the parallelograms ABCD, EBCF, be on the same base BC, and between the same parallels AF, BC.

**Then the parallelogram ABCD shall be equal to the parallelogram EBCF.**

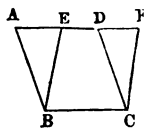
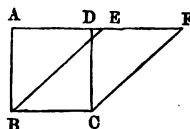


## DEMONSTRATION

If the sides AD, DF, of the parallelograms ABCD, DBCF, opposite to the base BC, be terminated in the same point D,

*it is plain that each of the parallelograms ABCD, DBCF, is double of the triangle BDC; (I. 34)*

**and therefore the parallelograms ABCD, DBCF, are equal to one another. (ax. 6.)**



But if the sides AD, EF, opposite to the base BC of the parallelograms ABCD, EBCF, be not terminated in the same point;

then, because ABCD is a parallelogram,

*therefore AD is equal to BC; (I. 34)*

for the same reason EF is equal to BC;

*wherefore AD is equal to EF; (ax. 1)*

and DE is common;

*therefore the whole, or remainder\* AE, is equal to the whole, or remainder DF; (ax. 2 or 3)*

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\* The words 'whole or remainder' are used to suit both figures (2 and 3), since, in figure 2, DE is *added* to AD and EF, which therefore gives the whole AE; and in figure 3, DE is *taken away* from AD and EF, which therefore leaves the remainder AE.

and AB is equal to DC. (I. 34.)

Wherefore, in the triangles EAB, FDC,

because the two EA, AB, are equal to the two FD, DC, each to each;  
and the exterior angle FDC is equal to the interior EAB, (I. 29)

therefore the base EB is equal to the base FC,

and the triangle EAB equal to the triangle FDC. (I. 4.)

Take the triangle FDC from the trapezium ABCF,  
and from the same trapezium take the triangle EAB;  
the remainders are therefore equal, (ax. 3)

that is, the parallelogram ABCD is equal to the parallelogram EBCF.

Therefore, parallelograms upon the same base, &c.

Q. E. D.

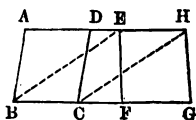
### PROP. XXXVI.—THEOREM.

*Parallelograms upon equal bases and between the same parallels are equal to one another.*

(References—Prop. I. 33, 34, 35; ax. 1.)

Let ABCD, EFGH be parallelograms upon equal bases BC, FG, and between the same parallels AH, BG.

Then the parallelogram ABCD shall be equal to the parallelogram EFGH.



#### CONSTRUCTION

From B to E, draw the straight line BE, and from C to H, the straight line CH.

#### DEMONSTRATION

Because BC is equal to FG, (hyp.) and FG to EH, (I. 34)

therefore BC is equal to EH; (ax. 1)

and they are parallels, and joined towards the same parts by the straight lines BE, CH; (hyp.)

but 'straight lines which join equal and parallel straight lines towards the same parts, are themselves equal and parallel;' (I. 33)

therefore EB, CH, are both equal and parallel;

wherefore EBCH is a parallelogram. (I. 34, def.)

Then because the parallelograms  $ABCD$ ,  $EBCH$ , are upon the same base  $BC$ ,

and between the same parallels  $BC$ ,  $AH$ ;

*therefore the parallelogram  $ABCD$  is equal to the parallelogram  $EBCH$ ;*

(I. 35)

for the like reason, the parallelogram  $EFGH$  is equal to the same  $EBCH$ ;

**therefore the parallelogram  $ABCD$  is equal to the parallelogram  $EFGH$ . (ax. 1.)**

Wherefore, parallelograms, &c.

Q. E. D.

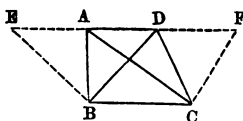
### PROP. XXXVII.—THEOREM.

*Triangles upon the same base, and between the same parallels, are equal to one another.*

(References—Prop. I. 31, 34, 35; ax. 7.)

Let the triangles  $ABC$ ,  $DBC$ , be on the same base  $BC$ , and between the same parallels  $AD$ ,  $BC$ .

**Then the triangle  $ABC$  shall be equal to the triangle  $DBC$ .**



#### CONSTRUCTION

Produce  $AD$  both ways to the points  $E$ ,  $F$ , and through  $B$  draw  $BE$  parallel to  $CA$ , and through  $C$  draw  $CF$  parallel to  $BD$ . (I. 31.)

#### DEMONSTRATION

Then, each of the figures  $EBCA$ ,  $DBCF$ , is a parallelogram; (I. 34, def.)

and because the parallelograms  $EBCA$ ,  $DBCF$ , are upon the same base  $BC$ ,

and between the same parallels  $BC$ ,  $EF$ ,

*therefore the parallelogram  $EBCA$  is equal to the parallelogram  $DBCF$ ;*

(I. 35)

and because the diameter  $AB$  bisects the parallelogram  $EBCA$ ;

*therefore the triangle  $ABC$  is the half of  $EBCA$ ; (I. 34)*

and because the diameter DC bisects the parallelogram DBCF,  
*therefore the triangle DBC is the half of DBCF;*  
 but 'things which are halves of the same are equal to one another,'  
 (ax. 7)  
*therefore the triangle ABC is equal to the triangle DBC.*  
 Wherefore, triangles, &c. Q. E. D.

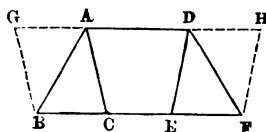
## PROP. XXXVIII.—THEOREM.

*Triangles upon equal bases and between the same parallels are equal to one another.*

(References — Prop. I. 31, 34, 36; ax. 7.)

Let the triangles ABC, DEF, be on the equal bases BC, EF, and between the same parallels AD, BF.

**Then the triangle ABC shall be equal to the triangle DEF.**



## CONSTRUCTION

Produce AD both ways to the points G, H,  
 and through B draw BG parallel to CA, and through F draw FH  
 parallel to ED. (I. 31.)

## DEMONSTRATION

Then, each of the figures GBCA, DEFH, is a parallelogram; (I. 34 def.)

and because they are upon equal bases BC, EF,  
 and between the same parallels BF, GH,

*therefore these parallelograms are equal to one another.* (I. 36.)

And because the diameter AB bisects the parallelograms GBCA,

*therefore the triangle ABC is the half of GBCA;* (I. 34)

and because the diameter DF bisects the parallelogram DEFH,

*therefore the triangle DEF is the half of DEFH;*

but 'things which are halves of the same are equal to one another,'  
 (ax. 7.)

**therefore the triangle ABC is equal to the triangle DEF.**

Wherefore, triangles, &c.

Q. E. D.



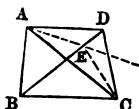
## PROP. XXXIX.—THEOREM.

*Equal triangles upon the same base and on the same side of it, are between the same parallels.*

(References — Prop. I. 31, 37 ; ax. 1.)

Let the equal triangles  $ABC$ ,  $DBC$ , be upon the same base  $BC$ , and on the same side of it.

**Then they shall be between the same parallels.**



## CONSTRUCTION

From  $A$  to  $D$ , draw the straight line  $AD$ .

Then  $AD$  is parallel to  $BC$ ,

For, if it be not, through the point  $A$  draw  $AE$  parallel to  $BC$ , (I. 31) and join  $EC$ .

## DEMONSTRATION

Then, because the triangles  $ABC$ ,  $EBC$ , are on the same base  $BC$ , and are assumed to be between the same parallels  $BC$ ,  $AE$ ;

*therefore the triangle  $ABC$  must be equal to the triangle  $EBC$  ; (I. 37)*

but the triangle  $ABC$  is equal to the triangle  $DBC$  ; (hyp.)

*therefore the triangle  $DBC$  must be equal to the triangle  $EBC$ , (ax. 1)*  
the greater to the less, which is impossible;

*therefore  $AE$  is not parallel to  $BC$ .*

In the same manner, it can be demonstrated that no other line but  $AD$  is parallel to  $BC$  ;

**therefore  $AD$  is parallel to  $BC$ .**

Wherefore, equal triangles upon, &c.

Q. E. D.

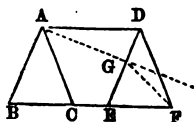
## PROP. XL.—THEOREM.

*Equal triangles upon equal bases in the same straight line and towards the same parts are between the same parallels.*

(References — Prop. I. 31, 38 ; ax. 1.)

Let the equal triangles  $ABC$ ,  $DEF$ , be upon the equal bases  $BC$ ,  $EF$ , in the same straight line  $BF$ , and towards the same parts.

**Then the triangles  $AEC$ ,  $DFE$ , shall be between the same parallels.**



## CONSTRUCTION

From A to D, draw the straight line AD;  
then AD is parallel to BF.

For, if it be not, through A draw AG parallel to BF, (I. 31) and join GF.

## DEMONSTRATION

Because the triangles ABC, GEF, are upon equal bases BC, EF,  
and are assumed to be between the same parallels BF, AG;

*therefore the triangle ABC must be equal to the triangle GEF; (I. 38)*

but the triangle ABC is equal to the triangle DEF, (hyp.)

*therefore also the triangle DEF must be equal to the triangle GEF,*  
(ax. 1)

the greater to the less, which is impossible;

*therefore AG is not parallel to BF.*

And in the same manner it can be demonstrated that there is no other  
parallel to it but AD;

**therefore AD is parallel to BF.**

Wherefore equal triangles, &c.

Q. E. D.

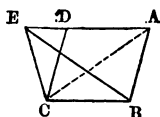
## PROP. XII.—THEOREM.

*If a parallelogram and a triangle be upon the same base and between the same parallels; the parallelogram shall be double of the triangle.*

(References — Prop. I. 34, 37; ax. 1.)

Let the parallelogram ABCD, and the triangle EBC be upon the same  
base BC, and between the same parallels BC, AE.

**Then the parallelogram ABCD shall be double of the  
triangle EBC.**



## CONSTRUCTION

From A to C draw the straight line AC

## DEMONSTRATION

Because the triangles ABC, ECB are upon the same base BC, and between the same parallels BC, AE,

*therefore the triangle ABC is equal to the triangle ECB; (I. 37)*

but the diameter AC bisects the parallelogram ABCD,

*therefore the parallelogram ABCD is double of the triangle ABC, (I. 34)*

**wherefore also ABCD is double of the triangle ECB. (ax. 1)**

Therefore if a parallelogram, &c.

Q. E. D.

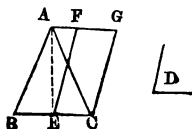
## PROP. XLII.— PROBLEM.

*To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.*

(References — Prop. I. 10, 23, 31, 38, 41; ax. 6; def. I. 34.)

Let ABC be the given triangle, and D the given rectilineal angle.

It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.



## CONSTRUCTION

Bisect BC in E, (I. 10) join AE,

and at the point E in the straight line EC, make the angle CEF equal to D; (I. 23)

through A draw AFG parallel to BC, and through C draw CG parallel to EF. (I. 31.)

**Then FECH is the parallelogram required.**

DEMONSTRATION

Because the base BE is equal to the base EC, and BC parallel to AG, therefore the triangle ABE is equal to the triangle AEC; (I. 38)

*and therefore the triangle ABC is double of the triangle AEC;*

but because the parallelogram FECH and the triangle AEC are on the same base EC, and between the same parallels EC, AG,

*therefore the parallelogram FECH is double of the triangle AEC;*  
(I. 41)

but 'things which are double of the same are equal to one another'  
(ax. 6)

**therefore the parallelogram FECH is equal to the triangle ABC,**

**and it has one of its angles CEF equal to the given angle D.**  
(constr.)

Wherefore there has been described a parallelogram FECH equal to a given triangle ABC, having one of its angles CEF equal to the given angle D.  
Q. E. F.

PROP. XLIII.—THEOREM.

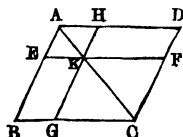
*The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.*

(References — Prop. I. 34 ; ax. 2, 3.)

Let ABCD be a parallelogram, of which the diameter is AC; and EH, FG, the parallelograms about AC, *that is, through which AC passes;*

and let BK, KD be the other parallelograms, which make up the whole figure ABCD, and are therefore called the complements.

**Then the complement BK shall be equal to the complement KD.**



DEMONSTRATION

Because ABCD is a parallelogram, and AC its diameter,

therefore the triangle  $ABC$  is equal to the triangle  $ADC$ ; (I. 34)  
 and because  $EKHA$  is a parallelogram, and  $AK$  its diameter,  
 therefore the triangle  $AEK$  is equal to the triangle  $AHK$ ;  
 and for the same reason,  
 the triangle  $KGC$  is equal to the triangle  $KFC$ .  
 And, because the triangle  $AEK$  is equal to the triangle  $AHK$ , and the  
 triangle  $KGC$  to  $KFC$ ;  
 therefore the two triangles  $AEK$ ,  $KGC$ , are equal to the two triangles,  
 $AHK$ ,  $KFC$ ; (ax. 2)  
 but the whole triangle  $ABC$  is equal to the whole triangle  $ADC$ ;  
 therefore the remaining complement  $BK$  is equal to the  
 remaining complement  $KD$ . (ax. 3.)  
 Wherefore, the complements, &c. Q. E. D.

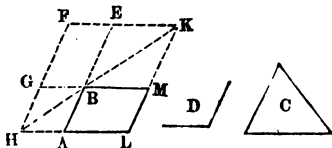
#### PROP. XLIV.— PROBLEM.

*To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.*

(References — Prop. I. 15, 29, 31, 42, 43; ax. 9. 12.)

Let  $AB$  be the given straight line, and  $C$  the given triangle, and  $D$  the given rectilineal angle.

It is required to apply to the straight line  $AB$ , a parallelogram equal to the triangle  $C$ , and having an angle equal to  $D$ .



#### CONSTRUCTION

Make the parallelogram  $BEFG$  equal to the triangle  $C$ , (I. 42)  
 and having the angle  $EBG$  equal to the angle  $D$ ,  
 so that  $BE$  be in the same straight line with  $AB$ ;  
 produce  $FG$  to  $H$ ,  
 and through  $A$  draw  $AH$  parallel to  $BG$  or  $EF$ , (I. 31) and join  $HB$ .

Then because the straight line  $HF$  falls upon the parallels  $AH$ ,  $EF$ ,  
 therefore the angles  $AHF$ ,  $HFE$ , are together equal to the two  
 right angles; (I. 29)

*wherefore the angles BHF, HFE, are less than two right angles; (ax. 9)*  
 but 'straight lines which with another straight line make the interior angles upon the same side less than two right angles, do meet if produced far enough;' (ax. 12)

*therefore HB, FE, shall meet, if produced;*

let them meet in K,

and through K draw KL parallel to EA or FH, (I. 31)

and produce HA, GB, to the points L, M.

**Then LB shall be the parallelogram required.**

#### DEMONSTRATION

Because HLKF is a parallelogram, of which the diameter is HK, and AG, ME, are the parallelograms about HK, also LB, BF, are the complements;

*therefore the complement LB is equal to the complement BF; (I. 43)*  
 but the complement BF is equal to the triangle C; (constr.)

**wherefore LB is equal to the triangle C.**

And because the angle GBE is equal to the angle ABM, (I. 15) and likewise to the angle D,

**therefore the angle ABM is equal to the angle D.**

Wherefore the parallelogram LB is applied to the straight line AB, and is equal to the triangle C, and has the angle ABM equal to the angle D. Q. E. F.

#### PROP. XLV.—PROBLEM.

*To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.*

(References—Prop. I. 14, 29, 30, 42, 44; ax. 1, 2; I. 34, def.)

Let ABCD be the given rectilineal figure, and E the given rectilineal angle.

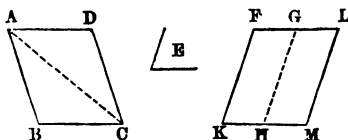
It is required to describe a parallelogram equal to ABCD, and having an angle equal to E.

#### CONSTRUCTION

Join AC, and describe the parallelogram FH equal to the triangle ABC, and having the angle HKF equal to the angle E; (I. 42)

and to the straight line GH apply the parallelogram GM equal to the triangle ADC, having the angle GHM equal to the angle E. (I. 44.)

**Then the figure FEML shall be the parallelogram required.**



## DEMONSTRATION

Because the angle E is equal to each of the angles FKH, GHM,  
*therefore the angle FKH is equal to the angle GHM;*  
 add to each of these the angle KHG;  
*therefore the angles FKH, KHG, are equal to the angles KHG, GHM;*  
 but FKH, KHG are equal to two right angles; (I. 29)  
*therefore also KHG, GHM, are equal to two right angles;*  
 and because at the point H in the straight line GH, the two straight  
 lines KH, HM, upon the opposite sides of it, make the adjacent angles  
 equal to two right angles,  
*therefore KH is in the same straight line with HM. (I. 14.)*

And because the straight line HG meets the parallels KM, FG,  
*therefore the alternate angles MHG, HGF, are equal; (I. 29)*  
 add to each of these the angle HGL;  
*therefore the angles MHG, HGL, are equal to the angles HGF, HGL;*  
 but the angles MHG, HGL, are equal to two right angles; (I. 29)  
*wherefore also the angles HGF, HGL, are equal to two right angles,*  
*and therefore FG is in the same straight line with GL. (I. 14.)*

And because KF is parallel to HG, and HG to ML,  
*therefore KF is parallel to ML; (I. 30)*

and KM, FL, are parallels;

*wherefore KFLM is a parallelogram; (I. 34, def.)*

and because the triangle ABC is equal to the parallelogram HF, and  
 the triangle ADC to the parallelogram GM;

**therefore the whole rectilineal figure ABCD is equal to  
 the whole parallelogram KFLM. (ax. 2.)**

Wherefore the parallelogram KFLM has been described equal to the  
 given rectilineal figure ABCD, having the angle FKM equal to the  
 angle E. Q. E. F.

Cor. From this it is manifest how to apply to a given right line a  
 parallelogram, which shall have an angle equal to a given rectilineal

angle, and shall be equal to a given rectilineal figure—viz., by applying to the given straight line a parallelogram equal to the first triangle ABC, and having an angle equal to the given angle.

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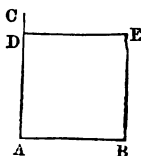
PROP. XLVI.—PROBLEM.

*To describe a square upon a given straight line.*

(References—Prop. I. 3, 11, 29, 31, 34; ax. 1, 3; def. 30.)

Let AB be the given straight line.

It is required to describe a square upon AB.



CONSTRUCTION

From the point A draw AC at right angles to AB, (I. 11)

and make AD equal to AB, (I. 3)

through the point D draw DE parallel to AB, (I. 31)

and through B draw BE parallel to AD.

**Then ADEB shall be the square required.**

DEMONSTRATION

Because DE is parallel to AB, and BE parallel to AD; (constr.)

therefore ADEB is a parallelogram; (I. 34, def.)

wherefore AB is equal to DE, and AD to BE; (I. 34)

but BA is equal to AD; (constr.)

therefore the four straight lines BA, AD, DE, EB, are equal to one another,

and the parallelogram ADEB is equilateral.

Likewise all its angles are right angles;

for, since the straight line AD meets the parallels AB, DE,

therefore the angles BAD, ADE are equal to two right angles; (I. 29)

but BAD is a right angle; (constr.)

therefore also ADE is a right angle; (ax. 3)

but 'the opposite angles of parallelograms are equal;' (I. 34)

therefore each of the opposite angles ABE, BED, is a right angle;



wherefore the parallelogram ADEB is rectangular.  
 And it has been demonstrated that it is equilateral;  
 therefore ADEB is a square,  
 and it is described upon the given straight line AB. Q. E. F.

Cor. Hence every parallelogram that has one right angle has all its angles right angles.

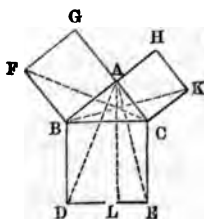
### PROP. XLVII.—THEOREM.

*In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.*

(References—Prop. I. 4, 14, 31, 41, 46; ax. 2, 4, 11; def. 30.)

Let ABC be a right-angled triangle, having the right angle BAC.

Then the square described upon the side BC, shall be equal to the squares described upon BA, AC.



#### CONSTRUCTION

On BC describe the square BDEC, (I. 46)  
 and on BA, AC, the squares GB, HC;  
 through A draw AL parallel to BD, or CE, (I. 31) and join AD, FC.

#### DEMONSTRATION

Then, because each of the angles BAC, BAG, is a right angle, (hyp., def. 30)  
 the two straight lines AC, AG, upon the opposite sides of AB, make with it at the point A, the adjacent angles equal to two right angles;  
 therefore CA is in the same straight line with AG; (I. 14)  
 for the same reason,

AB and AH are in the same straight line.

And because the angle DBC is equal to the angle FBA, each of them being a right angle, (ax. 11)

add to each the angle ABC,

*then the whole angle DBA is equal to the whole FBC. (ax. 2)*

Hence, in the two triangles ABD, FBC,

because the two sides AB, BD, are equal to the two FB, BC, each to each, (def. 30)

and the angle DBA equal to the angle FBC;

therefore the base AD is equal to the base FC,

*and the triangle ABD to the triangle FBC. (I. 4.)*

Again, because the parallelogram BL and the triangle ABD are upon the same base BD, and between the same parallels BD, AL;

*therefore the parallelogram BL is double of the triangle ABD; (I. 41)*

and because the square GB and the triangle FBC are upon the same base FB, and between the same parallels FB, GC;

*therefore the square GB is double of the triangle FBC;*

but the doubles of equals are equal to one another; (ax. 6)

*therefore the parallelogram BL is equal to the square GB.*

In the same manner, by joining AE, BK, it is proved that the parallelogram CL is equal to the square HC.

*Therefore the whole square BDEC is equal to the two squares GB, HC : (ax. 2.)*

and the square BDEC is described upon the straight line BC,

and the squares GB, HC, upon BA, AC;

**wherefore the square upon the side BC, is equal to the squares upon the sides BA, AC.**

Therefore, in any right-angled triangle, &c.

Q. E. D.

### PROP. XLVIII.—THEOREM

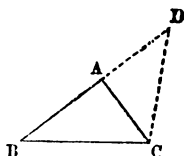
*If the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it;*

*then the angle contained by these two sides is a right angle.*

(References — Prop. I. 3, 8, 11, 47; ax. 1, 2.)

Let the square described upon BC, one of the sides of the triangle ABC, be equal to the squares upon the other sides BA, AC.

**Then the angle BAC shall be a right angle.**



## CONSTRUCTION

From the point A draw AD at right angles to AC, (I. 11) and make AD equal to BA, (I. 3) and join DC.

## DEMONSTRATION

Then, because DA is equal to AB,

*the square of DA is equal to the square of AB;*

to each of these equals add the square of AC,

*therefore the squares of DA, AC, are equal to the squares of BA, AC;*  
(ax. 2)

but because DAC is a right angle, (constr.)

*therefore the square of DC is equal to the squares of DA, AC; (I.*

*and the square of BC is equal to the squares of BA, AC; (hyp.)*

*therefore the square of DC is equal to the square of BC; (ax. 1*

*wherefore also the side DC is equal to the side BC.*

Hence, in the two triangles DAC, BAC,

because the side DA is equal to the side AB, (constr.)

and AC common to both triangles,

the two DA, AC, are equal to the two BA, AC, each to each,

and the base DC has been proved to be equal to the base BC;

*therefore the angle DAC is equal to the angle BAC; (I. 8)*

but DAC is a right angle; (constr.)

**therefore also BAC is a right angle.** (ax. 1.)

Therefore, if the square, &c.

Q. E.

## BOOK II.

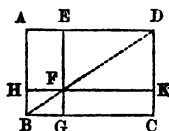
## DEFINITIONS.

## I.

EVERY right-angled parallelogram, or rectangle, is said to be contained by any two of the straight lines which contain one of the right angles.

## II.

In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon.



'Thus the parallelogram HG, together with the complements AF, FC, is the Gnomon, which is more briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the Gnomon.'

## PROP. I.—THEOREM.

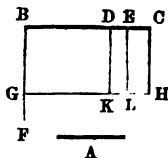
*If there be two straight lines, one of which is divided into any number of parts;*

*then the rectangle contained by the two straight lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.*

(References—Prop. I. 3, 11, 31, 34.)

Let  $A$  and  $BC$  be two straight lines; and let  $BC$  be divided into any number of parts in  $D$  and  $E$ .

**Then the rectangle contained by the straight lines  $A$  and  $BC$  is equal to the rectangle contained by  $A$  and  $BD$ , together with that contained by  $A$  and  $DE$ , and  $A$  and  $EC$ .**



## CONSTRUCTION

From the point  $B$ , draw  $BF$  at right angles to  $BC$ , (I. 11)  
and make  $BG$  equal to  $A$ ; (I. 3)  
through  $G$  draw  $GH$  parallel to  $BC$ ; (I. 31)  
and through the points  $D, E, C$ , draw  $DK, EL, CH$ , parallel to  $BG$ .

## DEMONSTRATION

Then the rectangle  $BH$  is equal to the rectangles  $BK, DL, EH$ ;  
but because  $BH$  is contained by  $GB, BC$ ,  
and  $GB$  is equal to  $A$ ; (constr.)

*therefore  $BH$  is contained by  $A$  and  $BC$ ,*  
and, because  $BK$  is contained by  $GB, BD$ , of which  $GB$  is equal to  $A$ ,  
*therefore  $BK$  is contained by  $A$  and  $BD$ ;*  
and because  $DL$  is contained by  $DK$  and  $DE$ ,  
and  $DK$ , that is,  $BG$ , (I. 34) is equal to  $A$ ;  
*therefore  $DL$  is contained by  $A$  and  $DE$ ;*  
and in like manner  $EH$  is contained by  $A$  and  $EC$ ;

**therefore the rectangle contained by  $A$  and  $BC$  is equal to the several rectangles contained by  $A$  and  $BD$ , by  $A$  and  $DE$ , and by  $A$  and  $EC$ .**

Wherefore, if there be two straight lines, &c.

Q E D.

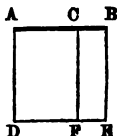
## PROP. II.—THEOREM.

*If a straight line be divided into any two parts;  
then the rectangles contained by the whole and each of the parts are together equal to the square of the whole line.*

(References—Prop. I. 31, 46.)

Let  $AB$  be divided into any two parts in  $C$ .

**Then the rectangle\* contained by  $AB$  and  $BC$ , together with the rectangle contained by  $AB$  and  $AC$ , shall be equal to the square of  $AB$ .**



#### CONSTRUCTION

Upon  $AB$  describe the square  $ADEB$ , (I. 46)  
and through  $C$  draw  $CF$  parallel to  $AD$  or  $BE$ . (I. 31.)

#### DEMONSTRATION

Then  $AE$  is equal to the rectangles  $AF$ ,  $CE$ .

But  $AE$  is the square of  $AB$ ;

*therefore the square of  $AB$  is equal to the rectangles  $AF$ ,  $CE$ .*

But  $AF$  is contained by  $DA$ ,  $AC$ , of which  $AD$  is equal to  $AB$ ,  
(def. 30)

*therefore  $AF$  is the rectangle contained by  $AB$ ,  $AC$ ;*

and  $CE$  is contained by  $AB$ ,  $BC$ , since  $BE$  is equal to  $AB$ ;

**therefore the rectangle contained by  $AB$ ,  $AC$ , together with the rectangle  $AB$ ,  $BC$ , is equal to the square of  $AB$ .**

If, therefore, a straight line, &c.

Q. E. D.

#### PROP. III.—THEOREM.

*If a straight line be divided into any two parts ;*

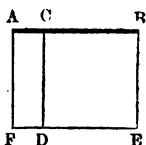
*then the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.*

(References—Prop. I. 31, 46.)

Let the straight line  $AB$  be divided into any two parts in the point  $C$ .

\* N.B. To avoid repeating the word *contained* too frequently, the rectangle contained by two straight lines,  $AB$  and  $AC$ , is sometimes simply called the rectangle  $AB$ ,  $AC$ .

**Then the rectangle AB, BC, shall be equal to the rectangle AC, CB, together with the square of CB.**



#### CONSTRUCTION

Upon BC describe the square CDEB; (I. 46)  
produce ED to F; and through A draw AF parallel to CD or BE.  
(I. 31.)

#### DEMONSTRATION

Then the rectangle AE is equal to the rectangles AD, CE.  
But AE is the rectangle contained by AB, BE, of which BE is equal to BC, (def. 30)

*therefore AE is the rectangle contained by AB, BC;*

and AD is contained by AC, CD, of which CD is equal to BC,

*therefore AD is contained by AC, CB;*

and CE is the square of BC;

**therefore the rectangle AB, BC, is equal to the rectangle AC, CB, together with the square of BC.**

If, therefore, a straight line be divided, &c.

Q. E. D.

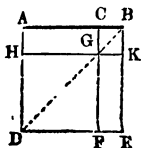
#### PROP. IV.—THEOREM.

*If a straight line be divided into any two parts;  
then the square of the whole line is equal to the squares of the two parts,  
together with twice the rectangle contained by the parts.*

(References—Prop. I. 5, 6, 29, 31, 34, 43, 46.)

Let the straight line AB be divided into any two parts in C.

**Then the square of AB shall be equal to the squares of AC, CB, together with twice the rectangle AC, CB.**



## CONSTRUCTION

Upon AB describe the square ADEB, (I. 46) and join DB;  
through C draw CGF parallel to BE or AD,  
and through G draw HGK parallel to AB or DE. (I. 31.)

## DEMONSTRATION

Then, because CF is parallel to AD, and BD falls upon them,  
*therefore the exterior angle BGC is equal to the interior and opposite angle ADB ; (I. 29)*  
but, because AB is equal to AD, being the sides of the square ADEB ;  
(def. 30)  
therefore, the angle ADB is equal to the angle CBG ; (I. 5)  
and the angle CGB is therefore equal to the angle CBG ; (ax. 1)  
*wherefore the side BC is equal to the side CG ; (I. 6)*  
but CB is equal also to GK, and CG to BK ; (I. 34)  
*therefore the figure CGKB is equilateral. (ax. 1.)*

It is likewise rectangular,  
for since CG is parallel to BK, and CB meets them,  
*therefore the angles KBC, GCB, are equal to two right angles ; (I. 29)*  
but KBC is a right angle ; (def. 30)  
wherefore GCB is a right angle,  
*and therefore also the opposite angles CGK, GKB, are right angles,*  
(I. 34)  
*wherefore the figure CGKB is rectangular ;*  
but it is also equilateral, as was demonstrated,  
*therefore CGKB is a square ;*  
and it is upon the side CB.

For the same reason, also,  
HF is a square,  
and it is upon the side HG, which is equal to AC ; (I. 34)  
*therefore HF, CK, are the squares of AC, CB.*

And, because the complement AG is equal to the complement GE,  
(I. 43)  
and that AG is the rectangle contained by AC, CB, since GC is equal to CB ; (def. 30)  
therefore also GE is equal to the rectangle AC, CB ;  
*wherefore AG, GE, are equal to twice the rectangle AC, CB ;*  
and HF, CK, are the squares of AC, CB ;



*therefore the four figures HF, CK, AG, GE, are equal to the squares of AC, CB, together with twice the rectangle AC, CB;*

*but HF, CK, AG, GE, make up the whole figure ADEB, which is the square of AB;*

*therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle AC, CB.*

Wherefore, if a straight line, &c.

Q. E. D.

Cor. From the demonstration, it is manifest that the parallelograms about the diameter of a square are likewise squares.

### PROP. V.—THEOREM.

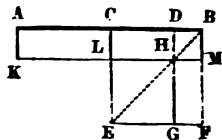
*If a straight line be divided into two equal parts, and also into two unequal parts;*

*then the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.*

(References—Prop. I. 31, 34, 36, 43, 46; II. 4, Cor.)

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D.

**Then the rectangle AD, DB, together with the square of CD, shall be equal to the square of CB.**



#### CONSTRUCTION

Upon CB describe the square CEFB, (I. 46) and join BE;  
through D draw DHG parallel to CE or BF; (I. 31)  
through H draw KLM parallel to CB or EF;  
and through A draw AK parallel to CL or BM.

#### DEMONSTRATION

Then, because the complement CH is equal to the complement HF  
(I. 43)  
to each of these add DM;

*therefore the whole CM is equal to the whole DF ;*  
 but CM is equal to AL, (I. 36) because AC is equal to CB ; (hyp.)  
*therefore also, AL is equal to DF ;*  
 to each of these equals add CH,  
*therefore the whole AH is equal to DF and CH.*

But AH is contained by AD, DB, since DH is equal to DB ;  
 and DF, together with CH, is the gnomon CMG ;

*therefore the gnomon CMG is equal to the rectangle AD, DB ; (ax. 1)*  
 to each of these equals add LG, which is equal to the square of CD,  
 (II. 4, Cor.) since CD is equal to LH ;

*therefore the gnomon CMG, together with LG, is equal to the rectangle*  
*AD, DB, together with the square of CD ;*

but the gnomon CMG and LG make up the whole figure CEFB, which  
 is the square of CB ;

**therefore the rectangle AD, DB, together with the square**  
**of CD, is equal to the square of CB.**

Wherefore, if a straight line, &c.

Q. E. D.

Cor. From this proposition it is manifest that the difference of the  
 squares of two unequal lines AC, CD, is equal to the rectangle con-  
 tained by their sum and difference.

### PROP. VI.—THEOREM.

*If a straight line be bisected, and produced to any point ;*  
*then the rectangle contained by the whole line thus produced, and the part*  
*of it produced, together with the square of half the line bisected, is equal to*  
*the square of the straight line which is made up of the half and the part*  
*produced.*

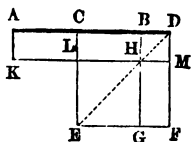
(References—Prop. I. 31, 36, 43, 46 ; II. 4, Cor.)

Let the straight line AB be bisected in C, and produced to D.

**Then the rectangle AD, DB, together with the square of**  
**BC, shall be equal to the square of CD.**

#### CONSTRUCTION

Upon CD describe the square CEFD, (I. 46) and join DE ;  
 through B draw BHG parallel to CE or DF ; (I. 31)  
 through H draw KLM parallel to AD or EF ;  
 and through A draw AK parallel to CL or DM.



## DEMONSTRATION

Because AC is equal to CB, (hyp.)

*therefore the rectangle AL is equal to CH, (I. 36)*

but CH is equal to HF; (I. 43)

*therefore also AL is equal to HF;*

to each of these add CM;

*therefore the whole AM is equal to the gnomon CMG.*

But AM is the rectangle contained by AD, DB, since DM is equal to DB; (II. 4, Cor.)

*therefore the gnomon CMG is equal to the rectangle AD, DB;*

add to each of these LG, which is equal to the square of CB;

*therefore the rectangle AD, DB, together with the square of CB, is equal to the gnomon CMG, and the figure LG;*

but the gnomon CMG and LG make up the whole figure CEFD, which is the square of CD;

**therefore the rectangle AD, DB, together with the square of CB, is equal to the square of CD.**

Wherefore, if a straight line, &c.

Q. E. D.

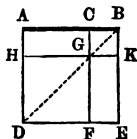
## PROP. VII.—THEOREM.

*If a straight line be divided into any two parts;  
then the squares of the whole line and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.*

(References — Prop. I. 31, 34, 43, 46; II. 4. Cor.)

Let the straight line AB be divided into any two parts in the point C.

**Then the squares of AB, BC, shall be equal to twice the rectangle AB, BC, together with the square of AC.**



## CONSTRUCTION

Upon AB describe the square ADEB, (I. 46) and join BD;  
 through C draw CGF parallel to AD or BE, (I. 31)  
 and through G draw HGK parallel to AB or DE.

## DEMONSTRATION

Then, because AG is equal to GE, (I. 43)

add to each of them CK;

therefore the whole AK is equal to the whole CE;

and therefore AK, CE, are double of AK;

but AK, CE, are the gnomon AKF, together with the square CK,

therefore the gnomon AKF, together with the square CK, is double of AK;

but twice the rectangle AB, BC, is double of AK, for BK is equal to BC; (II. 4, Cor.)

therefore the gnomon AKF, together with the square CK, is equal to twice the rectangle AB, BC;

to each of these equals add HF, which is equal to the square of AC;

therefore the gnomon AKF, together with the squares CK, HF, is equal to twice the rectangle AB, BC, and the square of AC;

but the gnomon AKF, together with the squares CK, HF, make up the whole figures ADEB and CK, which are the squares of AB and BC;

therefore the squares of AB and BC are equal to twice the rectangle AB, BC, together with the square of AC.

Wherefore, if a straight line, &c.

Q. E. D.

## PROP. VIII.—THEOREM.

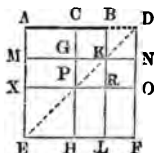
*If a straight line be divided into any two parts;  
 then four times the rectangle contained by the whole line, and one of the*

*parts, together with the square of the other part, is equal to the square of the straight line which is made up of the whole and that part.*

(References — Prop. I. 3, 31, 34, 36, 43, 46 ; II. 4, Cor.)

Let the straight line AB be divided into any two parts at point C.

**Then four times the rectangle AB, BC, together with the square of AC, shall be equal to the square of the straight line made up of AB and BC together.**



#### CONSTRUCTION

Produce AB to D, so that BD be equal to CB ; (I. 3)  
upon AD describe the square AEFD ; (I. 46)  
and construct two figures such as in the preceding.

#### DEMONSTRATION

Then, because CB is equal to BD, (constr.)  
and that CB is equal to GK, and BD to KN ; (I. 34)

*therefore GK is equal to KN ;*

for the same reason, PR is equal to RO ;

and because CB is equal to BD, and GK to KN,

*therefore the rectangle CK is equal to BN, and GR to RN ;*

but CK is equal to RN, because they are the complements of the parallelogram CO ; (I. 43)

*therefore also BN is equal to GR,*

*and therefore the four rectangles BN, CK, GR, RN, are equal to one another, and so are quadruple of one of them, CK.*

Again, because CB is equal to BD,  
and that BD is equal to BK, that is CG ; (I. 34)

and because CB is equal to GK, that is GP ;

*therefore CG is equal to GP ;*

and because CG is equal to GP, and PR to RO,

*the rectangle AG is equal to MP, and PL to RF ;*

but MP is equal to PL, because they are the complements of the parallelogram ML; (I. 43)

and AG is equal to RF;

*therefore the four rectangles AG, MP, PL, RF, are equal to one another, and so are quadruple of one of them, AG.*

And it was demonstrated that the four CK, BN, GR, and RN, are quadruple of CK;

*therefore the eight rectangles which contain the gnomon AOH, are quadruple of AK.*

And because AK is the rectangle contained by AB, BC, for BK is equal to BC,

*therefore four times the rectangle AB, BC, is quadruple of AK;*

but the gnomon AOH was demonstrated to be quadruple of AK;

*therefore four times the rectangle AB, BC, is equal to the gnomon AOH;*

to each of these add XH, which is equal to the square of AC; (II. 4, Cor.)

*therefore four times the rectangle AB, BC, together with the square of AC, is equal to the gnomon AOH, and the square XH;*

but the gnomon AOH and XH make up the whole figure AEFD, which is the square of AD;

**therefore four times the rectangle AB, BC, together with the square of AC, is equal to the square of AD,**

**that is, of AB and BC added together in one straight line.**

Wherefore, if a straight line, &c.

Q. E. D.

### PROP. IX.—THEOREM.

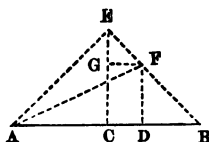
*If a straight line be divided into two equal and also into two unequal parts;*

*then the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.*

(Reference—Prop. I. 3, 5, 6, 11, 29, 31, 32, 34, 47.)

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts in the point D.

**Then the squares of AD, BD, shall be together double of the squares of AC, CD.**



## CONSTRUCTION

From the point C draw CE at right angles to AB, (I. 11) and make it equal to AC or CB, (I. 3) and join EA, EB; through D draw DF parallel to CE, (I. 31) meeting EB in F; and through F draw FG parallel to AB; and join AF.

## DEMONSTRATION

Then, because AC is equal to CE, (constr.)

*the angle EAC is equal to the angle AEC; (I. 5)*

and because the angle ACE is a right angle,

*the angles AEC, EAC, together make one right angle; (I. 32)*

and they are equal to one another;

*therefore each of the angles AEC, EAC, is half a right angle.*

For the same reason each of the angles CEB, EBC, is half a right angle;

*therefore the whole AEB is a right angle.*

And, because the angle GEF is half a right angle,

and EGF a right angle, for it is equal to the interior and opposite angle ECB, (I. 29)

*the remaining angle EFG is half a right angle;*

*therefore the angle GEF is equal to the angle EFG,*

*wherefore the side EG is equal to the side GF. (I. 6.)*

Again, because the angle at B is half a right angle,

and FDB a right angle, for it is equal to the interior and opposite angle ECB, (I. 29)

*the remaining angle BFD is half a right angle;*

*therefore the angle at B is equal to the angle BFD,*

*wherefore the side DF is equal to the side DB. (I. 6.)*

And because AC is equal to CE,

*the square of AC is equal to the square of CE.*

*therefore the squares of AC, CE, are double of the square of AC;*

but because ACE is a right angle

the square of EA is equal to the squares of AC, CE, (I. 47)

*therefore the square of EA is double of the square of AC.*

Again, because EG is equal to GF,

the square of EG is equal to the square of GF,

*therefore the squares of EG, GF, are double of the square of GF ;*

but the square of EF is equal to the squares of EG, GF ; (I. 47)

*therefore the square of EF is double of the square of GF ;*

and GF is equal to CD ; (I. 34)

*therefore the square of EF is double of the square of CD ;*

but the square of EA is likewise double of the square of AC ;

*therefore the squares of EA, EF, are double of the squares of AC, CD.*

And because AEF is a right angle,

the square of AF is equal to the squares of AE, EF, (I. 47)

*therefore the square of AF is double of the squares of AC, CD ;*

but the squares of AD, DF, are equal to the square of AF, because ADF is a right angle, (I. 47)

*therefore the squares of AD, DF, are double of the squares of AC, CD ;*

and DF is equal to DB ;

**therefore the squares of AD, DB, are double of the squares of AC, CD.**

Therefore if a straight line, &c.

Q. E. D.

#### PROP. X.—THEOREM.

*If a straight line be bisected, and produced to any point ;*

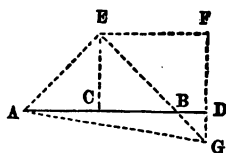
*then the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.*

(References—Prop. I. 5, 6, 11, 15, 29, 31, 32, 34, 46 Cor. 47.)

Let the straight line AB be bisected in C, and produced to the point D.

**Then the squares of AD, DB, shall be double of the squares of AC, CD.**





## CONSTRUCTION

From the point C draw CE at right angles to AB; (I. 11)  
make it equal to AC or CB, (I. 3) and join AE, EB;  
through E draw EF parallel to AB, (I. 31)  
and through D draw DF parallel to CE.

Then because the straight line EF meets the parallels EC, FD,  
the angles CEF, EFD, are equal to two right angles; (I. 29)  
*therefore the angles BEF, EFD, are less than two right angles;*

but 'straight lines which with another straight line make the interior  
angles upon the same side less than two right angles, do meet if pro-  
duced far enough;' (ax. 12)

*therefore EB, FD, will meet, if produced towards B and D;*

let them meet in G, and join AG.

## DEMONSTRATION

Then, because AC is equal to CE,

*the angle CEA is equal to the angle EAC; (I. 5)*

and the angle ACE is a right angle,

*therefore each of the angles CEA, EAC, is half a right angle. (I. 32.)*

For the same reason each of the angles CEB, EBC, is half a right  
angle;

*therefore the whole AEB is a right angle.*

And because EBC is half a right angle,

*the angle DBG which is vertically opposite is also half a right angle,*  
(I. 15.)

but BDG is a right angle, because it is equal to the alternate angle  
DCE; (I. 29.)

*therefore the remaining angle DGB is half a right angle, and is  
therefore equal to the angle DBG;*

*wherefore also the side BD is equal to the side DG. (I. 6.)*

Again, because EGF is half a right angle,

and that the angle at F is a right angle, because it is equal to the  
opposite angle ECD, (I. 34.)

the angle FEG is half a right angle, and therefore equal to the angle EGF;

*wherefore also the side GF is equal to the side FE.*

And because EC is equal to CA,

the square of EC is equal to the square of CA;

*therefore the squares of EC, CA, are double of the square of CA;*

but the square of EA is equal to the squares of EC, CA; (I. 47)

*therefore the square of EA is double of the square of AC.*

Again, because GF is equal to EF,

the square of GF is equal to the square of EF;

*and therefore the squares of GF, FE, are double of the square of EF;*

but the square of EG is equal to the squares of GF, EF; (I. 47)

*therefore the square of EG is double of the square of EF;*

and EF is equal to CD; (I. 34)

*wherefore the square of EG is double of the square of CD;*

But it was demonstrated that the square of EA is double of the square of AC;

*therefore the squares of AE, EG, are double of the squares of AC, CD;*

but the square of AG is equal to the squares of AE, EG; (I. 47)

*therefore the square of AG is double of the squares of AC, CD;*

but the square of AG is also equal to the squares of AD, DG; (I. 47)

*therefore the squares of AD, DG, are double of the squares of AC, CD;*

but DG is equal to DB;

**therefore the squares of AD, DB, are double of the squares of AC, CD.**

Wherefore, if a straight line, &c,

Q. E. D.

### PROP. XI.—PROBLEM,

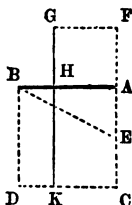
*To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.*

(References—Prop. I. 3, 10, 46, 47; II. 6.)

Let AB be the given straight line.

It is required to divide AB into two parts, so that the rectangle

contained by the whole and one of the parts shall be equal to the square of the other part.



#### CONSTRUCTION

Upon AB describe the square ABDC; (I. 46)  
 bisect AC in E, (I. 10) and join BE;  
 produce CA to F, and make EF equal to EB; (I. 3)  
 and upon AF describe the square FGHA. (I. 46.)

**Then AB shall be divided in H, so that the rectangle BH, is equal to the square of AH.**

Produce GH to K.

#### DEMONSTRATION

Then, because the straight line AC is bisected in E, and produced to the point F,

*the rectangle CF, FA, together with the square of AE, is equal to the square of EF; (II. 6)*

but EF is equal to EB;

*therefore the rectangle CF, FA, together with the square of AE, is equal to the square of EB;*

and the squares of BA, AE, are equal to the square of EB, because the angle EAB is a right angle;

*therefore the rectangle CF, FA, together with the square of AE, is equal to the squares of BA, AE;*

take away the square of AE, which is common to both;

*therefore the remaining rectangle CF, FA, is equal to the square of BA*

But the figure FK is the rectangle contained by CF, FA, since it is equal to FG; (def. 30)

and AD is the square of AB;

*therefore the figure FK is equal to AD;*

take away the common part AK,

*and the remainder FH is equal to the remainder HD;*

but HD is the rectangle contained by AB, BH, for AB is equal to BD ;  
and FH is the square of AH ;

**therefore the rectangle AB, BH, is equal to the square of AH.**

Wherefore the straight line AB is divided in H, so that the rectangle  
AB, BH, is equal to the square of AH. Q. E. F.

### PROP. XII.—THEOREM.

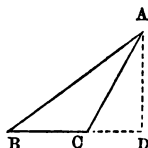
*In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced ;*

*then the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.*

(References — Prop. I. 12, 47 ; II. 4.)

Let ABC be an obtuse-angle triangle, having the obtuse angle ACB ;  
and from the point A let AD be drawn perpendicular to BC produced.  
(I. 12.)

**Then the square of AB shall be greater than the squares  
of AC, CB, by twice the rectangle BC, CD.**



### DEMONSTRATION

Because the straight line BD is divided into two parts in the point C,  
*the square of BD is equal to the squares of BC, CD, and twice the rectangle BC, CD ;* (II. 4)

to each of these equals add the square of DA ;

*then the squares of BD, DA, are equal to the squares of BC, CD, DA,  
and twice the rectangles BC, CD ;*

but the square of BA is equal to the squares of BD, DA, (I. 47) because  
the angle at D is a right angle ;

and the square of CA is equal to the squares of CD, DA ; (I. 47)

*therefore the square of BA is equal to the squares of BC, CA, and twice the rectangle BC, CD ;*

**that is, the square of BA is greater than the squares of BC, CA, by twice the rectangle BC, CD.**

Wherefore, in obtuse-angled triangles, &c.

Q. E. D.

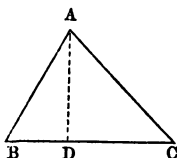
### PROP. XIII.—THEOREM.

*In every triangle, the square of the side subtending either of the acute angles, is less than the squares of the sides containing that angle, by twice the rectangle contained by either of those sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle, and the acute angle.*

(References — Prop. I. 12, 16, 47 ; II. 3, 7, 12.)

Let ABC be any triangle, and the angle at B one of its acute angles ; and upon BC, one of the sides containing it, let fall the perpendicular AD from the opposite angle. (I. 12.)

**Then the square of AC opposite to the angle B, shall be less than the squares of CB, BA, by twice the rectangle CB, BD.**



#### DEMONSTRATION

*First, Let AD fall within the triangle ABC.*

Then, because the straight line CB is divided into two parts in the point D,

*the squares of CB, BD, are equal to twice the rectangle contained by CB, BD, and the square of DC ; (II. 7)*

to each of these equals add the square of AD ;

*therefore the squares of CB, BD, DA, are equal to twice the rectangle CB, BD, and the squares of AD, DC ;*

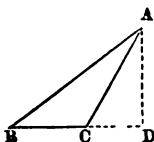
but the square of AB is equal to the squares of BD, DA, (I. 47) because the angle BDA is a right angle;

and the square of AC is equal to the squares of AD, DC;

*therefore the squares of CB, BA, are equal to the square of AC, and twice the rectangle CB, BD;*

**that is, the square of AC alone is less than the squares of CB, BA, by twice the rectangle CB, BD.**

*Secondly, Let AD fall without the triangle ABC.*



Then, because the angle at D is a right angle, the angle ACB is greater than a right angle; (I. 16)

*and therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle BC, CD; (II. 12)*

to each of these equals add the square of BC,

*then the squares of AB, BC, are equal to the square of AC, and twice the square of BC, and twice the rectangle BC, CD;*

but because BD is divided into two parts in C,

*the rectangle DB, BC, is equal to the rectangle BC, CD, and the square of BC; (II. 3)*

and the doubles of these are equal;

*that is, twice the rectangle DB, BC, is equal to twice the rectangle BC, CD, and twice the square of BC;*

*therefore the squares of AB, BC, are equal to the square of AC, and twice the rectangle DB, BC;*

**wherefore the square of AC alone is less than the squares of AB, BC, by twice the rectangle DB, BC.**

*Lastly, let the side AC be perpendicular to BC.*



## DEMONSTRATION

Then BC is the straight line between the perpendicular and the acute angle at B ;

and it is manifest that the squares of AB, BC, are equal to the square of AC, and twice the square of BC.

Therefore, in every triangle, &c.

Q. E. D.

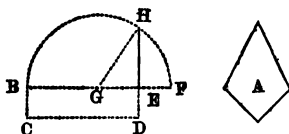
## PROP. XIV.—PROBLEM.

*To describe a square that shall be equal to a given rectilineal figure.*

(References — Prop. I. 3, 10, 45, 47 ; II. 5.)

Let A be the given rectilineal figure.

It is required to describe a square that shall be equal to A.



## CONSTRUCTION

Describe the rectangular parallelogram BCDE equal to the rectilineal figure A. (I. 45.)

If, then, the sides of it, BE, ED, are equal to one another, it is a square, and what was required is now done.

But if they are not equal, produce one of them, BE to F, and make EF equal to ED ;

bisect BF in G, (I. 10) and from the centre G, at the distance GB, or GF, describe the semicircle BHF, produce DE to meet the circumference in H.

**Then the square described upon EH shall be equal to the given rectilineal figure A.**

Join GH.

## DEMONSTRATION

Because the straight line BF is divided into two equal parts in the point G, and into two unequal parts in E,

*the rectangle BE, EF, together with the square of EG, is equal to the square of GF; (II. 5)*

but GF is equal to GH; (def. 15)

*therefore the rectangle BE, EF, together with the square of EG, is equal to the square of GH;*

but the squares of HE, EG, are equal to the square of GH; (I. 47)

*therefore the rectangle BE, EF, together with the square of EG, is equal to the squares of HE, EG;*

take away the square of EG, which is common to both,

*and the remaining rectangle BE, EF, is equal to the square of EH.*

But the rectangle contained by BE, EF, is the parallelogram BD, because EF is equal to ED;

*therefore BD is equal to the square of EH;*

but BD is equal to the rectilineal figure A;

**therefore the rectilineal figure A is equal to the square of EH.**

Wherefore a square has been made equal to the given rectilineal figure A — viz., the square described upon EH. Q. E. F.

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# BOOK III.

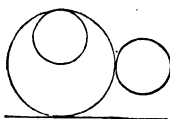
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## DEFINITIONS.

### I.

**EQUAL** circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

### II.



A straight line is said to touch a circle when it meets the circle, and being produced does not cut it.

### III.

Circles are said to touch each other which meet, but do not cut each other.



### IV.

Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.

### V.

And the straight line on which the greater perpendicular falls, is said to be farther from the centre.

### VI.



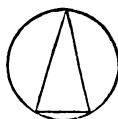
A segment of a circle is a figure contained by a straight line, and the circumference which it cuts off.

VII.

The angle of a segment is that which is contained by the straight line and the circumference.

VIII.

An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment, to the extremities of the straight line which is the base of the segment.



IX.

An angle is said to insist or stand upon the circumference intercepted between the straight lines that contain the angle.

X.

A sector of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.



XI.

Similar segments of circles are those in which the angles are equal, or which contain equal angles.



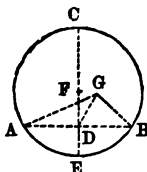
## PROP. I.—PROBLEM.

*To find the centre of a given circle.*

(References—Prop. I. 8, 10, 11.)

Let ABC be the given circle.

It is required to find its centre.



## CONSTRUCTION

Draw within the circle any straight line AB, and bisect AB in D;  
(I. 10)

from the point D draw DC at right angles to AB (I. 11), produce it to E, and bisect CE in F.

**Then the point F shall be the centre of the circle ABC.**

## DEMONSTRATION

For, if it be not, let, if possible, G be the centre, and join GA, GD, GB.

Then, because DA is equal to DB, (constr.) and DG common to the two triangles ADG, BDG,

the two sides AD, DG, are equal to the two BD, DG, each to each;  
and the base GA is assumed to be equal to the base GB, (I. def. 15)  
because they are drawn from the centre G;

*therefore the angle ADG must be equal to the angle GDB; (I. 8)*

but 'when a straight line standing upon another straight line makes the adjacent angles equal to one another, each of the angles is a right angle;' (I. def. 10)

*therefore the angle GDB must be a right angle;*

but FDB is a right angle; (constr.)

*wherefore the angle FDB must be equal to the angle GDB,*

the greater to the less, which is impossible;

*therefore G is not the centre of the circle ABC.*

*In the same manner it can be shown, that no other point but F is the centre;*

therefore **F** is the centre of the circle **ABC**,  
which was to be found.

Cor.—From this it is manifest, that if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

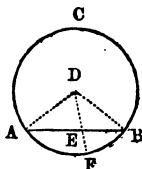
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PROP. II.—THEOREM.

*If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.*

(References—Prop. I. 5, 16, 19 ; III. 1.)

Let **ABC** be a circle, and **A**, **B**, any two points in the circumference.  
Then the straight line drawn from **A** to **B** shall fall within the circle.



CONSTRUCTION

Find **D** the centre of the circle **ABC**, (III. 1) and join **AD**, **DB** ; in the circumference **AB** take any point **F** ; join **DF**, and let it cut the straight line **AB** in **E**.

DEMONSTRATION

Because **DA** is equal to **DB**, (I. def. 15)

*the angle **DAB** is equal to the angle **DBA** ; (I. 5)*

and because **AE**, a side of the triangle **DAE**, is produced to **B**,

*the exterior angle **DEB** is greater than the interior and opposite angle **DAE** ; (I. 16)*

but the angle **DAE** was proved to be equal to the angle **DBE** ;

*therefore the angle **DEB** is also greater than the angle **DBE** ;*

but to the greater angle the greater side is opposite ; (I. 19)

*therefore **DB** is greater than **DE** ;*

but **DB** is equal to **DF** ; (I. def. 15)

therefore  $DF$  is greater than  $DE$ ,  
 wherefore the point  $E$  lies within the circle.

In the same manner it may be proved that every point in  $AB$  lies within the circle;

**therefore the straight line  $AB$  lies within the circle.**

Wherefore, if any two points, &c.

Q. E. D.

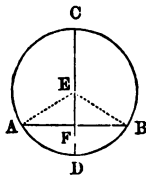
### PROP. III.—THEOREM.

*If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and conversely, if it cut it at right angles, it shall bisect it.*

(References—Prop. I. 5, 8, 26; III. 1.)

Let  $ABC$  be a circle, and let  $CD$ , a straight line drawn through the centre, bisect any straight line  $AB$ , which does not pass through the centre, in the point  $F$ .

**Then  $CD$  shall cut  $AB$  at right angles.**



#### CONSTRUCTION

Take  $E$  the centre of the circle (III. 1), and join  $EA$ ,  $EB$ .

#### DEMONSTRATION

Then, because  $AF$  is equal to  $FB$  (hyp.), and  $FE$  common to the two triangles  $AFE$ ,  $BFE$ ,  
 there are two sides in the one equal to two sides in the other, each to each;

and the base  $EA$  is equal to the base  $EB$ ; (I. def. 16)

*therefore the angle  $AFE$  is equal to the angle  $BFE$ ; (I. 8)*

but 'when a straight line standing upon another straight line makes the adjacent angles equal to one another, each of them is a right angle; (I. def. 10)

*therefore each of the angles AFE, BFE, is a right angle;*  
**wherefore the straight line CD, drawn through the centre,**  
**bisecting another AB that does not pass through the**  
**centre, cuts the same at right angles.**

But let CD cut AB at right angles.

**Then CD shall also bisect it, that is, AF shall be equal to FB.**

The same construction being made;  
 because EA, EB, from the centre are equal to one another,

*the angle EAF is equal to the angle EBF; (I. 5)*

and the right angle AFE is equal to the right angle BFE; (I. def. 10)

hence, in the two triangles EAF, EBF,  
 there are two angles in the one equal to two angles in the other, each to each;

and the side EF, which is opposite to one of the equal angles in each, is common to both;

*therefore the other sides are equal; (I. 26)*

**therefore AF is equal to FB.**

Wherefore, if a straight line, &c.

Q. E. D.

#### PROP. IV.—THEOREM.

*If in a circle two straight lines cut one another which do not both pass through the centre, they do not bisect each other.*

(References — Prop. III. 1, 3.)

Let ABCD be a circle, and AC, BD, two straight lines in it which cut one another in the point E, and do not both pass through the centre.

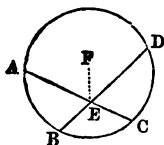
**Then AC, BD, shall not bisect one another.**

#### CONSTRUCTION

For, if it be possible, let AE be equal to EC, and BE to ED.

If one of the lines pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre:

but if neither of them pass through the centre,  
 take F the centre of the circle, (III. 1) and join EF.



## DEMONSTRATION

Because FE, a straight line through the centre, is assumed to bisect another AC which does not pass through the centre, therefore it must cut it at right angles; (III. 3)  
*wherefore FEA must be a right angle.*

Again, because the straight line FE is assumed to bisect the straight line BD which does not pass through the centre, therefore it must cut it at right angles; (III. 3)  
*wherefore FEB must be a right angle;*

but FEA was shown to be a right angle;

*therefore FEA must be equal to the angle FEB,*

the less to the greater, which is impossible;

**therefore AC, BD, do not bisect one another.**

Wherefore, if in a circle, &c.

Q. E. D.

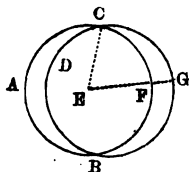
## PROP. V.—THEOREM.

*If two circles cut one another, they shall not have the same centre.*

(References — I. def. 15.)

Let the two circles ABC, CDG, cut one another in the points B, C.

**Then they shall not have the same centre.**



## CONSTRUCTION

For, if it be possible, let E be their centre; join EC, and draw any straight line EFG meeting them in F and G.

## DEMONSTRATION

And because E is assumed to be the centre of the circle ABC,

*EC must be equal to EF; (I. def. 15)*

again, because E is assumed to be the centre of the circle CDG,

*EC must be equal to EG;*

but EC was shown to be equal to EF;

*therefore EF must be equal to EG;*

the less to the greater, which is impossible;

**therefore E is not the centre of the circles ABC, CDG.**

Wherefore, if two circles, &c.

Q. E. D.

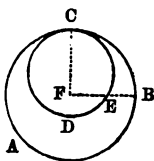
## PROP. VI.—THEOREM.

*If one circle touch another internally, they shall not have the same centre.*

(References — I. def. 15.)

Let the circle CDE, touch the circle ABC internally in the point C.

**Then they shall not have the same centre.**



## CONSTRUCTION

For, if they can, let the centre be F; join FC, and draw any straight line FEB meeting them in E and B.

## DEMONSTRATION

Because F is assumed to be the centre of the circle ABC,

*therefore CF must be equal to FB; (I. def. 15)*

also, because F is assumed to be the centre of the circle CDE,

*therefore CF must be equal to FE;*

but CF was shown to be equal to FB;

1



therefore  $FE$  must be equal to  $FB$ ; (ax. 1.)

the less to the greater, which is impossible;

wherefore  $F$  is not the centre of the circles  $ABC$ ,  $CDE$ .

Therefore, if two circles, &c.

Q. E. D.

### PROP. VII.—THEOREM.

*If any point be taken in the diameter of a circle, which is not the centre;*

*then of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the line which passes through the centre is always greater than one more remote; and from the same point there can be drawn only two straight lines to the circumference that are equal to each other, one upon each side of the shortest line.*

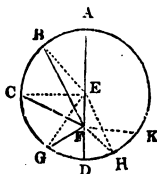
(References—Prop. I. 4, 20, 23, 24.)

Let  $ABCD$  be a circle, and  $AD$  its diameter, in which let any point  $F$  be taken which is not the centre; let the centre be  $E$ .

Then of all the straight lines  $FB$ ,  $FC$ ,  $FG$ , &c., that can be drawn from  $F$  to the circumference,  $FA$  which passes through the centre, shall be the greatest;

and  $FD$ , the other part of the diameter  $AD$ , shall be the least;

and of the others,  $FB$ , the nearer to  $FA$ , shall be greater than  $FC$ , the more remote; and  $FC$  greater than  $FG$ .



CONSTRUCTION

Join  $BE$ ,  $CE$ ,  $GE$ .

## DEMONSTRATION

Because two sides of a triangle are greater than the third,

*therefore BE, EF, are greater than BF; (I. 20)*

but AE is equal to EB; (I. def. 15)

*therefore AE, EF, that is AF, is greater than BF.*

Again, in the triangles BEF, CEF,

because BE is equal to CE, and EF common to both,

the two sides BE, EF, are equal to the two CE, EF, each to each;

but the angle BEF is greater than the angle CEF; (ax. 9)

*therefore the base BF is greater than the base CF; (I. 24)*

for the same reason, CF is greater than GF.

Again, because GF, FE, are greater than EG, (I. 20) and EG is equal to ED;

*therefore GF, FE, are greater than ED;*

take away the common part FE,

*and the remainder GF is greater than the remainder FD; (ax. 5)*

**therefore FA is the greatest of all the straight lines from**

**F to the circumference,**

**and FD is the least;**

**and BF is greater than CF, and CF than GF.**

Also there can be drawn only two equal straight lines from the point F to the circumference, one upon each side of the shortest line FD.

## CONSTRUCTION

At the point E, in the straight line EF, make the angle FEH equal to the angle GEF, (I. 23) and join FH.

## DEMONSTRATION

Then in the triangles GEF, HEF,

because GE is equal to EH, (I. def. 15) and EF common to both,

the two sides GE, EF, are equal to the two HE, EF, each to each;

and the angle GEF is equal to the angle HEF; (constr.)

*therefore the base FG is equal to the base FH; (I. 4)*

but besides FH, no straight line can be drawn from F to the circumference equal to FG;

for, if there can, let it be FK;

and because FK is assumed to be equal to FG, and FG is equal to FH,

*therefore FK must be equal to FH; (ax. 1)*

that is, a line nearer to that which passes through the centre, is equal to one which is more remote;  
 which has been proved to be impossible.  
 Therefore, if any point be taken, &c. Q. E. D.

PROP. VIII.—THEOREM.

*If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre;*

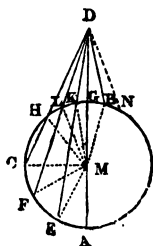
*then of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to the one passing through the centre, is always greater than one more remote; but of those which fall upon the convex circumference, the least is that between the point without the circle and the diameter; and of the rest that which is nearer to the least is always less than one more remote; and only two equal straight lines can be drawn from the same point to the circumference, one upon each side of the line which passes through the centre.*

(References — Prop. I. 4, 20, 21, 23, 24; III. 1.)

Let ABC be a circle, and D any point without it, from which let the straight lines DA, DE, DF, DC, be drawn to the circumference whereof DA passes through the centre.

**Then, of those which fall upon the concave part of the circumference AEFC, AD which passes through the centre shall be the greatest; and the nearer to it is always greater than the more remote, viz., DE than DF, and DF than DC.**

**but of those which fall upon the convex circumference HLEK, the least shall be DG between the point D and the diameter AG; and the nearer to it is always less than the more remote, viz., DK than DL, and DL than DH.**



## CONSTRUCTION

Take  $M$  the centre of the circle  $ABC$ , (III. 1) and join  $ME$ ,  $MF$ ,  $MC$ ,  $MK$ ,  $ML$ ,  $MH$ .

## DEMONSTRATION

And because  $AM$  is equal to  $EM$ ,  
add  $MD$  to each,

*therefore  $AD$  is equal to  $EM$ ,  $MD$  ; (ax. 2)*

but  $EM$ ,  $MD$ , are greater than  $ED$  ; (I. 20)

*therefore also  $AD$  is greater than  $ED$ .*

Again, in the triangles  $EMD$ ,  $FMD$ ,  
because  $ME$  is equal to  $MF$ , and  $MD$  common to both,  
 $EM$ ,  $MD$ , are equal to  $FM$ ,  $MD$ , each to each,  
but the angle  $EMD$  is greater than the angle  $FMD$  ; (ax. 9)

*therefore the base  $ED$  is greater than the base  $FD$ . (I. 24.)*

In like manner it may be shown that  $FD$  is greater than  $CD$  ;

**therefore  $DA$  is the greatest line,**

**and  $DB$  is greater than  $DF$ , and  $DF$  than  $DC$ .**

And because  $MK$ ,  $KD$ , are greater than  $MD$ , (I. 20)

and  $MK$  is equal to  $MG$ , (I. def. 15)

the remainder  $KD$  is greater than the remainder  $GD$  ; (ax. 5)

*that is,  $GD$  is less than  $KD$  ;*

and because  $MK$ ,  $DK$ , are drawn to the point  $K$  within the triangle  $MLD$ , from  $M$ ,  $D$ , the extremities of its side  $MD$ ,

*therefore  $MK$ ,  $KD$  are less than  $ML$ ,  $LD$  ; (I. 21)*

but  $MK$  is equal to  $ML$  ; (I. def. 15)

*therefore the remainder  $DK$  is less than the remainder  $DL$ . (ax. 5.)*

In like manner it may be shown that  $DL$  is less than  $DH$  ;

**therefore  $DG$  is the least, and  $DK$  less than  $DL$ , and  $DL$  than  $DH$ .**

Also, there can be drawn only two equal straight lines from the point  $D$  to the circumference, one upon each side of the line through the centre.

## CONSTRUCTION

At the point  $M$ , in the straight line  $MD$ , make the angle  $DMB$  equal to the angle  $DMK$ , (I. 23) and join  $DB$ .

## DEMONSTRATION

Then, in the triangles  $KMD$ ,  $BMD$ ,

because MK is equal to MB, and MD common to both,  
the two sides KM, MD, are equal to the two BM, MD, each to each;  
and the angle KMD is equal to the angle BMD; (constr.)

*therefore the base DK is equal to the base DB ; (I. 4)*

but besides DB there can be no straight line drawn from D to the circumference equal to DK.

For, if there can, let it be DN;

and because DK is assumed to be equal to DN, and DK has been shown to be equal to DB,

*therefore DB must be equal to DN,*

that is, a line nearer to the least is equal to one more remote,  
which has been proved to be impossible.

If, therefore, any point, &c.

Q. E. D.

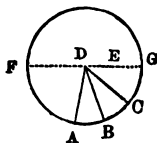
### PROP. IX.—THEOREM.

*If a point be taken within a circle, from which there can be drawn more than two equal straight lines to the circumference; then that point is the centre of the circle.*

(References—III. 7.)

Let the point D be taken within the circle ABC, from which to the circumference there can be drawn more than two equal straight lines,  
viz. DA, DB, DC.

**Then the point D shall be the centre of the circle.**



#### CONSTRUCTION

For, if not, let E be the centre; join DE, and produce it to the circumference in F, G.

#### DEMONSTRATION

Then FG is a diameter of the circle ABC ; (I. def. 17)

and because in FG, the diameter of the circle ABC, there is taken the point D, not the centre,

*therefore DG must be the greatest line from the point D to the circumference, and DC greater than DB, and DB than DA ; (III. 7)*

but these lines are likewise equal, (hyp.) which is impossible.

*therefore E is not the centre of the circle ABC.*

In like manner it may be demonstrated that no other point but D is the centre ;

**therefore D is the centre of the circle ABC.**

**Wherefore, if a point be taken, &c.**

**Q. E. D.**

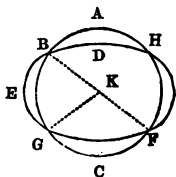
**PROP. X.—THEOREM.**

*One circumference of a circle cannot cut another in more than two points.*

(References—III. 3, 5, 9.)

**Let the two circles ABC, DEF, intersect one another.**

**Then their circumferences cannot cut each other in more than two points.**



## CONSTRUCTION

If it be possible, let the circumference FAB cut the circumference DEF in more than two points, viz. in B, G, F;  
take the centre K of the circle ABC, (III. 3) and join KB, KG, KF.

## DEMONSTRATION

And because within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal straight lines KB, KG, KF,

*the point K must be the centre of the circle DEF, (III. 9)*

but **K** is also the centre of the circle **ABC**;

therefore the point  $K$  must be the centre of two circles that cut one another.

which is impossible. (III. 5.)

**Therefore one circumference of a circle cannot cut another in more than two points.** Q. E. D.

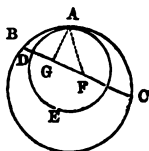
**Q. E. D.**

## PROP. XI.—THEOREM.

*If one circle touch another internally in any point, the straight line which joins their centres, being produced, shall pass through that point.*

(References — Prop. I. 20.)

Let the circle ADE touch the circle ABC internally in the point A.  
Then the straight line which joins their centres, being produced, shall pass through the point A.



## CONSTRUCTION

For, if not, let it fall otherwise, if possible, as FGDB; let F be the centre of the circle ABC, and G the centre of ADE. Join AF and AG.

## DEMONSTRATION

Because two sides of a triangle are together greater than the third side  
therefore AG, GF, are greater than FA; (I. 20)

but FA is assumed to be equal to FB, (I. def. 15) both being from the same centre;

therefore AG, GF, must be greater than FB;

take away the common part FG;

then the remainder AG must be greater than the remainder GB;

but AG is assumed to be equal to GD; (I. def. 15)

therefore GD must be greater than GB,

the less than the greater, which is impossible;

therefore the straight line which joins the centres, being produced, cannot fall otherwise than upon the point A.

that is, it must pass through the point of contact of the two circles.

Therefore, if two circles, &c.

Q. E. D.

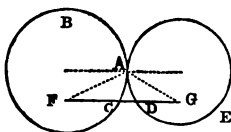
## PROP. XII.—THEOREM.

*If two circles touch each other externally, the straight line which joins their centres shall pass through the point of contact.*

(References — Prop. I. 20.)

Let the two circles ABC, ADE, touch each other externally in the point A.

**Then the straight line which joins their centres, shall pass through the point of contact A.**



## CONSTRUCTION

For, if not, let, if possible, F and G be the centres, and the line joining them pass otherwise as FCDG; and join FA, AG.

## DEMONSTRATION

Then, because F is assumed to be the centre of the circle ABC,  
*therefore AF must be equal to FC;*

also, because G is assumed to be the centre of the circle ADE,  
*therefore AG must be equal to GD;*

and therefore FA, AG, must be equal to FC, DG;

*wherefore the whole FG must be greater than FA, AG;*

but FG is also less than FA, AG; (I. 20) which is impossible.

*therefore the straight line which joins the centres of the circles, shall not pass otherwise than through A;*

**that is, it must pass through the point of contact of the two circles.**

Therefore, if two circles, &c.

Q. E. D.

## PROP. XIII.—THEOREM.

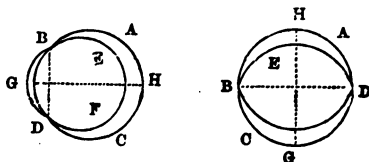
*One circle cannot touch another in more points than one, whether it touches it on the inside or the outside.*

(References — Prop. I. 10, 11; III. 1, 2, 11; Cor. 2, 11.)



*First.* Let the circle EBF touch the circle ABC internally in the point B.

**Then EBF cannot touch ABC in any other point.**



#### CONSTRUCTION

If it be possible, let EBF touch ABC in another point D ;  
join BD, and draw GH bisecting BD at right angles. (I. 10, 11)

#### DEMONSTRATION

Because the points B, D, are assumed to be in the circumference of each of the circles,

*the straight line BD must fall within each of the circles EBF, ABC ;*  
(III. 2)

and their centres are in the straight line GH, which bisects BD at right angles ; (III. 1, Cor.)

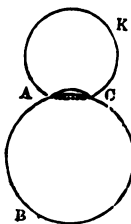
*therefore GH must pass through the point of contact ;* (III. 11)

but GH does not pass through the point of contact, because the points B, D, are without the straight line GH ; which is absurd ;

**therefore the circle EBF cannot touch the circle ABC on the inside in more points than one.**

*Secondly.* Let the circle ACK touch the circle ABC externally in the point A.

**Then ACK cannot touch ABC in any other point.**



#### CONSTRUCTION

If it be possible, let ACK touch ABC in another point C. Join AC.

## DEMONSTRATION

Then, because the points A, C, are assumed to be in the circumference of the circle ACK,

*the straight line AC must fall within the circle ACK; (III. 2)*

but the circle ACK is without the circle ABC; (hyp.)

*therefore the straight line AC must be without the circle ABC;*

but, because the points A, C, are assumed to be in the circumference of the circle ABC,

*the straight line AC must be within the circle; (III. 2)*

which is absurd;

**therefore the circle ACK cannot touch the circle ABC on the outside in more than one point.**

And it has been shown that one circle cannot touch another on the inside in more points than one.

Wherefore one circle, &c.

Q. E. D.

## PROP. XIV.—THEOREM.

*Equal straight lines in a circle are equally distant from the centre; and conversely, those which are equally distant from the centre, are equal to one another.*

(References — Prop. I. 12, 47; III. 1, 3, def. 4.)

Let the straight lines AB, CD, in the circle ABDC, be equal to one another.

**Then they shall be equally distant from the centre.**

## CONSTRUCTION

Take E the centre of the circle ABDC, (III. 1)

and from it draw EF, EG, perpendiculars to AB, CD, (I. 12) and join EA, EC.

## DEMONSTRATION

Then, because the straight line EF passing through the centre, cuts the straight line AB, which does not pass through the centre, at right angles, it also bisects it; (III. 3)

*therefore AF is equal to FB, and AB double of AF;*

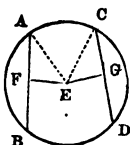
for the same reason

*CD is double of CG;*

but AB is equal to CD; (hyp.)

*therefore AF is equal to CG. (ax. 7.)*

F



And because AE is equal to EC, (I. def. 15)

• the square of AE is equal to the square of EC ;

but the squares of AF, FE, are equal to the square of AE, because the angle AFE is a right angle ; (I. 47)

and for the same reason, the squares of EG, GC, are equal to the square of EC ;

therefore the squares of AF, FE, are equal to the squares of CG, GE ;  
of which the square of AF is equal to the square of CG, because AF is equal to CG ;

therefore the remaining square of EF is equal to the remaining square of EG, (ax. 3)

and therefore the straight line EF is equal to EG ;

but 'straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal ;'  
(III. def. 4)

therefore AB, CD, are equally distant from the centre E.

Next, let the straight lines AB, CD, be equally distant from the centre, that is, let FE be equal to EG.

**Then AB shall be equal to CD.**

The same construction being made, it may be demonstrated as before,

that AB is double of AF, and CD double of CG,

and that the squares of EF, FA, are equal to the squares of EG, GC ;

of which the square of FE is equal to the square of EG, because FE is equal to EG ; (hyp.)

therefore the remaining square of AF is equal to the remaining square of CG ; (ax. 3)

and therefore the straight line AF is equal to CG ;

but AB was shown to be double of AF, and CB double of CG ;

wherefore AB is equal to CD.

Therefore, equal straight lines, &c.

Q. E. D.

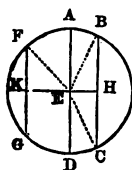
## PROP. XV.—THEOREM.

*The diameter is the greatest straight line in a circle, and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.*

(References—Prop. I. 12, 20; III. def. 5.)

Let ABCD be a circle, of which the diameter is AD, and the centre E; and let BC be nearer to the centre than FG.

**Then AD shall be greater than any straight line BC, which is not a diameter, and BC shall be greater than FG.**



## CONSTRUCTION

From the centre E draw EH, EK, perpendiculars to BC, FG, (I. 12) and join EB, EC, EF.

## DEMONSTRATION

Then, because AE is equal to EB, and ED to EC, (I. def. 15)

*therefore AD is equal to EB, EC; (ax. 2)*

but EB, EC, are greater than BC; (I. 20)

**wherefore also AD is greater than BC.**

And because BC is nearer to the centre than FG, (hyp.)

*therefore EH is less than EK; (III. def. 5)*

but, as was demonstrated in the preceding proposition,

*BC is double of BH, and FG double of FK,*

*and the squares of EH, HB, are equal to the squares of EK, KF,*

but the square of EH is less than the square of EK, because EH is less than EK;

*therefore the square of BH is greater than the square of FK,*

*and the straight line BH greater than FK;*

**and therefore BC is greater than FG.**

Next, let BC be greater than FG.

**Then BC shall be nearer to the centre than FG, that EH shall be less than EK.**

The same construction being made, because BC is greater than FG, therefore BH is greater than KF ;

*and the squares of BH, HE, are equal to the squares of FK, KE,*

but the square of BH is greater than the square of FK, because BH greater than FK ;

*therefore the square of EH is less than the square of EK,*

*and the straight line EH less than EK.*

**and therefore BC is nearer to the centre than F**  
(III. def. 5.)

Wherefore, the diameter, &c.

Q. E. D.

### PROP. XVI.—THEOREM.

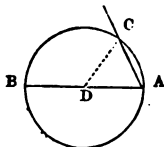
*The straight line which is drawn at right angles to the diameter of circle, from the extremity of it, falls without the circle ;*

*and no straight line can be drawn from the extremity between the straight line and the circumference, which does not cut the circle, or, which is the same thing, no straight line can make so great an acute angle with the diameter at its extremity, or so small an angle with the straight line which is at right angles to it, as not to cut the circle.*

(References—Prop. I. 5, 12, 17, 19 ; III. 2, def. 2.)

Let ABC be a circle, the centre of which is D, and the diameter AB.

**Then the straight line drawn at right angles to AB from its extremity A, shall fall without the circle.**



#### CONSTRUCTION

For, if it does not, let it fall, if possible, *within* the circle, as AC ; and draw DC to the point C, where it meets the circumference.

## DEMONSTRATION

Because DA is equal to DC, (I. def. 15)

*the angle DAC is equal to the angle ACD; (I. 5)*

but DAC is a right angle, (hyp.)

therefore ACD must be a right angle,

*and therefore the angles DAC, ACD, must be equal to two right angles;*

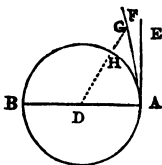
which is impossible; (I. 17)

*therefore the straight line drawn from A at right angles to BA, does not fall within the circle.*

In the same manner it may be demonstrated that it does not fall upon the circumference;

**therefore the straight line from A at right angles to BA, must fall without the circle, as AE.**

**Also, between the straight line AE and the circumference, no straight line can be drawn from the point A which does not cut the circle.**



## CONSTRUCTION

For, if possible, let FA be between them,

and from the point D draw DG perpendicular to FA, (I. 12)

and let it meet the circumference in H.

## DEMONSTRATION

Because AGD is assumed to be a right angle, and DAG less than a right angle; (I. 17)

*therefore DA must be greater than DG; (I. 19)*

but DA is equal to DH; (I. def. 15)

*therefore DH must be greater than DG,*

the less than the greater, which is impossible;

**therefore no straight line can be drawn from the point A, between AB and the circumference, which does not cut the circle;**

or, which amounts to the same thing, however great an acute angle a straight line makes with the diameter at the point A, or however

small an angle it makes with AE, the circumference shall pass between that straight line and the perpendicular AE.

‘And this is all that is to be understood, when in the Greek text, and in translations from it, the angle of the semicircle is said to be greater than any acute rectilinear angle, and the remaining angle less than any rectilinear angle.’

Q. E. D.

**Q. E. D.**

Cor.— From this it is manifest, that the straight line which is drawn at right angles to the diameter of a circle, from the extremity of it, touches the circle ; (III. def. 2) and that it touches it only in one point, because, if it did meet the circle in two, it would fall within it. (III. 2.)

'Also it is evident that there can be but one straight line which touches the circle in the same point.'

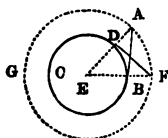
### PROP. XVII.—PROBLEM.

*To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.*

(References — Prop. I. 4, 11; III. 1, 16, cor.)

**First, let A be a given point without the given circle BCD.**

It is required to draw a straight line from A which shall touch the circle.



## CONSTRUCTION

Find the centre E of the circle, (III. 1) and join AE;  
and from the centre E, at the distance AE, describe the circle AFG;  
from the point D draw FD at right angles to AE; (I. 11) and join  
EBF, AB.

**Then AB shall touch the circle BCD.**

## DEMONSTRATION

Because E is the centre of the circles BCD, AFG;  
therefore EA is equal to EF, and ED to EB; (I. def. 15)  
hence, in the two triangles AEB, FED,

because the two sides AE, EB, are equal to the two FE, ED, each to each ;

and they contain the angle at E common to both triangles ;

therefore, the base DF is equal to the base AB,

and the triangle FED to the triangle AEB,

and the other angles to the other angles ;

therefore the angle EBA is equal to the angle EDF ; (I. 4)

but EDF is a right angle, (constr.)

wherefore EBA is a right angle ; (ax. 1)

and EB is drawn from the centre ;

but 'a straight line drawn from the extremity of a diameter, at right angles to it, touches the circle ;' (III. 16, cor.)

therefore AB touches the circle,

and it is drawn from the given point A.

But if the given point be in the circumference of the circle, as the point D,

draw DE to the centre E, and DF at right angles to DE ;

then DF touches the circle. (III. 16, cor.)

Q. E. F.

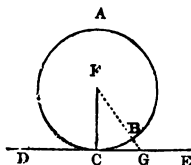
### PROP. XVIII.—THEOREM.

*If a straight line touch a circle, the straight line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.*

(References—Prop. I. 17, 19 ; III. 1.)

Let the straight line DE touch the circle ABC in the point C ;  
take the centre F, (III. 1) and draw the straight line FC.

**Then FC shall be perpendicular to DE.**



#### CONSTRUCTION

For, if it be not, from the point F draw FBG perpendicular to DE.



## DEMONSTRATION

And because  $FGC$  is assumed to be a right angle,

*therefore  $GCF$  must be an acute angle ; (I. 17)*

and to the greater angle the greater side is opposite ; (I. 19)

*therefore  $FC$  must be greater than  $FG$ ,*

but  $FC$  is equal to  $FB$ , (i. def. 15)

*therefore  $FB$  must be greater than  $FG$ ,*

the less than the greater, which is impossible;

*wherefore  $FG$  is not perpendicular to  $DE$ .*

In the same manner it may be shown, that no other is perpendicular to it besides  $FC$ ;

**that is,  $FC$  is perpendicular to  $DE$ .**

Therefore, if a straight line, &c.

Q. E. D.

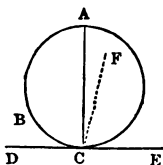
## PROP. XIX.—THEOREM.

*If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.*

(References — Prop. III. 18.)

Let the straight line  $DE$  touch the circle  $ABC$  in  $C$ ; and from  $C$  let  $CA$  be drawn at right angles to  $DE$ .

**Then the centre of the circle shall be in  $CA$ .**



## CONSTRUCTION

For, if not, let  $F$  be the centre, if possible, and join  $CF$ .

## DEMONSTRATION

Because  $DE$  touches the circle  $ABC$ , and  $FC$  is drawn from the assumed centre to the point of contact.

*therefore  $FC$  must be perpendicular to  $DE$ ; (III. 18)*

*and therefore  $FCE$  must be a right angle;*

but  $\angle ACE$  is a right angle, (hyp.)

therefore the angle  $\angle FCE$  must be equal to the angle  $\angle ACE$ , (ax. 1)  
the less to the greater, which is impossible.

wherefore  $F$  is not the centre of the circle  $ABC$ .

In the same manner, it may be shown that no other point which is not in  $CA$ , is the centre;

**that is, the centre of the circle is in  $CA$ .**

Therefore, if a straight line, &c.

Q. E. D.

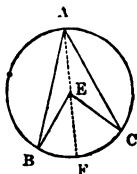
### PROP. XX.—THEOREM.

*The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.*

(References—Prop. I. 5, 32.)

Let  $ABC$  be a circle, and  $\angle BEC$  an angle at the centre, and  $\angle BAC$  an angle at the circumference, which have the same circumference  $BC$  for their base.

**Then the angle  $\angle BEC$  shall be double of the angle  $\angle BAC$ .**



*First, let the centre  $E$  of the circle be within the angle  $\angle BAC$ .*

#### CONSTRUCTION

Join  $AE$ , and produce it to the circumference in  $F$ .

#### DEMONSTRATION

Because  $EA$  is equal to  $EB$ ,

the angle  $\angle EAB$  is equal to the angle  $\angle EBA$ , (I. 5)

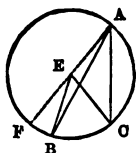
therefore the angles  $\angle EAB, \angle EBA$  are double of the angle  $\angle EAB$ ;

but the angle  $\angle BEF$  is equal to the angles  $\angle EAB, \angle EBA$ ; (I. 32)

therefore also the angle  $\angle BEF$  is double of the angle  $\angle EAB$ ;

for the same reason, the angle  $FEC$  is double of the angle  $EAC$  ;  
 therefore the whole angle  $BEC$  is double of the whole  
 angle  $BAC$ .

Secondly, let the centre  $E$  of the circle be *without* the angle  $BAC$ .



#### CONSTRUCTION

Join  $AE$ , and produce it to meet the circumference in  $F$ .

#### DEMONSTRATION

It may be demonstrated, as in the first case,  
*that the angle  $FEC$  is double of the angle  $FAC$ ,*  
*and that  $FEB$ , a part of  $FEC$ , is double of  $FAB$ , a part of  $FAC$  ;*  
 therefore the remaining angle  $BEC$  is double of the re-  
 maining angle  $BAC$ .

Therefore, the angle at the centre, &c.

Q. E. D.

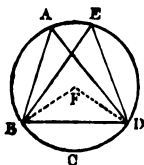
#### PROP. XXI.—THEOREM.

*The angles in the same segment of a circle are equal to one another.*

(References—Prop. III. 1, 20.)

Let  $ABCD$  be a circle, and  $BAD$ ,  $BED$ , angles in the same segment  $BAED$ .

Then the angles  $BAD$ ,  $BED$ , shall be equal to one another.



First, let the segment  $BAED$  be greater than a semicircle

## CONSTRUCTION

Take  $F$  the centre of the circle  $ABCD$ , (III. 1) and join  $BF$ ,  $FD$ .

## DEMONSTRATION

And, because the angle  $BFD$  is at the centre, and the angle  $BAD$  at the circumference, and that they have the same part of the circumference, viz.  $BCD$  for their base;

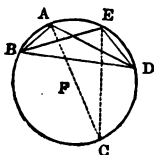
*therefore the angle  $BFD$  is double of the angle  $BAD$ ; (III. 20)*

for the same reason,

*the angle  $BFD$  is double of the angle  $BED$ ;*

**therefore the angle  $BAD$  is equal to the angle  $BED$ .**  
(ax. 7.)

*Next, let the segment  $BAED$  be not greater than a semicircle.*



## CONSTRUCTION

Draw  $AF$  to the centre, and produce it to  $C$ , and join  $CE$ .

## DEMONSTRATION

Then the segment  $BADC$  is greater than a semicircle,

*therefore by the first case, the angles  $BAC$ ,  $BEC$ , in it, are equal;*

for the same reason, because  $CBED$  is greater than a semicircle,

*the angles  $CAD$ ,  $CED$ , are equal;*

**therefore the whole angle  $BAD$  is equal to the whole angle  $BED$ .** (ax. 2.)

Wherefore, the angles in the same segment, &c.

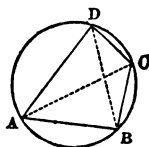
Q. E. D.

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 PROP. XXII.—THEOREM.

*The opposite angles of any quadrilateral figure inscribed in a circle, are together equal to two right angles.*

(References—Prop. I. 32; III. 21.)

Let  $ABCD$  be a quadrilateral figure in the circle  $ABCD$ .**Then any two of its opposite angles shall be together equal to two right angles.**

## CONSTRUCTION

Draw the diagonals  $AC$  and  $DB$ .

## DEMONSTRATION

Because the three angles of every triangle are equal to two right angles, (I. 32)

*the three angles of the triangle  $CAB$ , viz. the angles  $CAB$ ,  $ABC$ ,  $BCA$ , are equal to two right angles;*but the angle  $CAB$  is equal to the angle  $CDB$ , (III. 21) because they are in the same segment  $CDAB$ ;and the angle  $ACB$  is equal to the angle  $ADB$ , because they are in the same segment  $ADCB$ ;*therefore the whole angle  $ADC$  is equal to the angles  $CAB$ ,  $ACB$ ;*to each of these equals add the angle  $ABC$ ;*therefore the three angles  $ABC$ ,  $CAB$ ,  $BCA$ , are equal to the two angles  $ABC$ ,  $ADC$ ; (ax. 2)*but  $ABC$ ,  $CAB$ ,  $BCA$ , are equal to two right angles;**therefore also the angles  $ABC$ ,  $ADC$ , are equal to two right angles.**

In the same manner, it may be shown that

**the angles  $BAD$ ,  $DCB$ , are equal to two right angles.**

Therefore, the opposite angles, &amp;c.

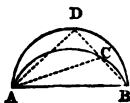
Q. E. D.

## PROP. XXIII.—THEOREM.

*Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.*

(References — Prop. I. 16 ; III. 10, def. 11.)

If it be possible, let the two similar segments of circles, viz. ACB, ADB, be upon the same side of the same straight line AB, not coinciding with one another.



#### CONSTRUCTION

Then, because the circle ACB cuts the circle ADB, in the two points A, B,

they cannot cut one another in any other point ; (III. 10)

*therefore one of the segments ACB, ADB, must fall within the other.*

Let ACB fall within ADB, and draw the straight line BCD, and join CA, DA.

#### DEMONSTRATION

Because the segment ACB is assumed to be similar to the segment ADB, (hyp.)

and that similar segments of circles contain equal angles ; (III. def. 11)

*the angle ACB must be equal to the angle ADB,*

the exterior to the interior, which is impossible ; (I. 16)

Therefore there cannot be two similar segments of circles upon the same side of the same line, which do not coincide. Q. E. D.

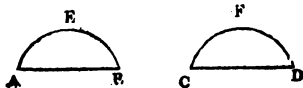
#### PROP. XXIV.—THEOREM.

*Similar segments of circles upon equal straight lines are equal to one another.*

(References — Prop. III. 23.)

Let AEB, CFD, be similar segments of circles upon the equal straight lines AB, CD.

**Then the segment AEB shall be equal to the segment CFD.**



## DEMONSTRATION

For if the segment AEB be applied to the segment CFD, so the point A may be on C, and the straight line AB on CD; then, because AB is equal to CD,  
*the point B shall coincide with the point D;*  
 and the straight line AB coinciding with CD,  
*the segment AEB coincides with the segment CFD, (III. 23)*  
 and therefore the segment AEB is equal to the segment CFD. (ax. 8.)

Wherefore, similar segments, &c.

Q. E. I

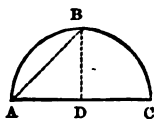
## PROP. XXV.—PROBLEM.

*A segment of a circle being given, to describe the circle of which the segment.*

(References — Prop. I. 4, 6, 10, 11, 23; III. 9.)

Let ABC be the given segment of a circle.

It is required to describe the circle of which it is the segment.



## CONSTRUCTION

Bisect AC in D, (I. 10) and from the point D draw DB at right angles to AC; (I. 11) and join AB.

*First, let the angles ABD, BAD, be equal to one another.*

**Then D shall be the centre of the circle required.**

## DEMONSTRATION

Because the angle ABD is equal to the angle BAD, (hyp.)

*therefore DA is equal to DB; (I. 6)*

but DA is equal to DC, (constr.)

*therefore DB is equal to DC,*

*and the three straight lines DA, DB, DC, are all equal;*

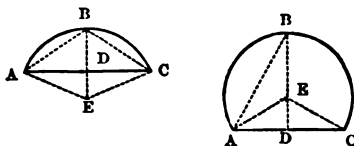
and because from the point D more than two equal straight lines DA, DB, DC, can be drawn to the circumference ABC,

**therefore D is the centre of the circle.** (III. 9.)

Hence, if with the centre D, at the distance of any of the three, DA, DB, DC, a circle be described,

*it will pass through the other two points, and be the circle required.*

*Secondly*, let the angles ABD, BAD, be not equal to one another.



#### CONSTRUCTION

At the point A, in the straight line BA, make the angle BAE equal to the angle ABD, (I. 23)

produce BD, if necessary, to E, and join EC.

**Then E shall be the centre of the circle required.**

#### DEMONSTRATION

Because the angle ABE is equal to the angle BAE, (constr.)

*the straight line BE is equal to EA ; (I. 6)*

and in the triangles ADE, CDE,

because AD is equal to DC, and DE common to both,

the two sides AD, DE, are equal to the two CD, DE, each to each ;

and the angle ADE is equal to the angle CDE, for each of them is a right angle ; (constr.)

*therefore the base AE is equal to the base EC ; (I. 4)*

but AE was shown to be equal to EB,

wherefore also EB is equal to EC ; (ax. 1)

*and the three straight lines EA, EB, EC, are therefore equal to one another ;*

and because from the point E, more than two equal straight lines, EA, EB, EC, can be drawn to the circumference ABC,

**therefore E is the centre of the circle.** (III. 9.)

Hence, if with the centre E, at the distance of any of the three AE, EB, EC, a circle be described,

*it will pass through the other two points, and be the circle required.*



In the *first case*, it is evident that, because the centre  $D$  is in  $AC$ ,  
*therefore the segment  $ABC$  is a semicircle.*

In the *second case*, if the angle  $ABD$  be greater than the angle  
 $BAD$ , the centre  $E$  falls without the segment  $ABC$ ,  
*which is therefore less than a semicircle ;*

but if the angle  $ABD$  be less than  $BAD$ , the centre  $E$  falls within the  
segment  $ABC$ ,  
*which is therefore greater than a semicircle.*

Wherefore a segment of a circle being given, the circle is described of  
which it is a segment. Q. E. F.

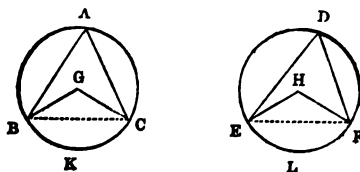
### PROP. XXVL—THEOREM.

*In equal circles, equal angles stand upon equal circumferences, whether  
they be at the centres or at the circumferences.*

(References—Prop. I. 4 ; III. 24, def. 1, 11.)

Let  $ABC$ ,  $DEF$ , be equal circles, having the equal angles  $BGC$ ,  
 $EHF$ , at their centres, and  $BAC$ ,  $EDF$ , at their circumferences.

**Then the circumference  $BKC$  shall be equal to the cir-  
cumference  $ELF$ .**



### CONSTRUCTION

Join  $BC$  and  $EF$ .

### DEMONSTRATION

Because the circles  $ABC$ ,  $DEF$  are equal, the straight lines drawn from  
their centres are equal ; (III. def. 1)

*therefore the two sides  $BG$ ,  $GC$ , are equal to the two  $EH$ ,  $HF$ , each to  
each ;*

*and the angle at  $G$  is equal to the angle at  $H$  ; (hyp.)*

*therefore the base BC is equal to the base EF. (I. 4.)*

And because the angle at A is equal to the angle at D, (hyp.)

*the segment BAC is similar to the segment EDF; (III. def. 11)*

and they are upon equal straight lines BC, EF;

but similar segments of circles upon equal straight lines are equal to one another; (III. 24)

*therefore the segment BAC is equal to the segment EDF;*

but the whole circle ABC is equal to the whole DEF; (hyp.)

*therefore the remaining segment BKC is equal to the remaining segment ELF, (ax. 3)*

**and the circumference BKC to the circumference ELF.**

Wherefore, in equal circles, &c.

Q. E. D.

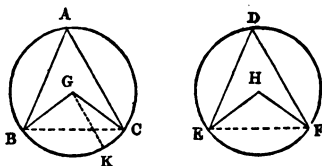
#### PROP. XXVII.—THEOREM.

*In equal circles, the angles which stand upon equal circumferences are equal to one another, whether they be at the centres or at the circumferences.*

(References—Prop. I. 23; III. 20, 26.)

Let the angles BGC, EHF, at the centres, and BAC, EDF, at the circumferences of the equal circles ABC, DEF, stand upon equal circumferences BC, EF.

**Then the angle BGC shall be equal to the angle EHF, and the angle BAC to the angle EDF.**



If the angle BGC be equal to the angle EHF,

*it is manifest that the angle BAC is also equal to EDF. (III. 20, and ax. 7.)*

But, if not, one of them is greater than the other; let BGC be the greater.

#### CONSTRUCTION

At the point G, in the straight line BG, make the angle BGK equal to the angle EHF. (I. 23.)

## DEMONSTRATION

Because equal angles stand upon equal circumferences when they are at the centre; (III. 26)

*therefore the circumference BK must be equal to the circumference EF;*  
but EF is equal to BC; (hyp.)

*therefore also BK must be equal to BC, (ax. 1)*  
the less to the greater, which is impossible;

*therefore the angle BGC is not unequal to the angle EHF;*

**that is, the angle BGC is equal to the angle EHF.**

And the angle at A is half of the angle BGC, (III. 20)

and the angle at D is half of the angle EHF;

**therefore the angle at A is equal to the angle at D.**  
(ax. 7.)

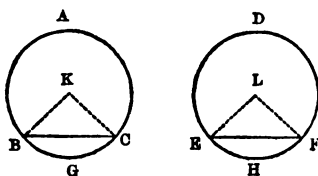
## PROP. XXVIII.—THEOREM.

*In equal circles, equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less.*

(References—Prop. I. 8; III. 1, 26, def. 1.)

Let ABC, DEF, be equal circles, and BC, EF, equal straight lines in them, which cut off the two greater circumferences BAC, EDF, and the two less, BGC, EHF.

**Then the greater circumference BAC shall be equal to the greater EDF, and the less BGC to the less EHF.**



## CONSTRUCTION

Take K, L, the centres of the circles, (III. 1) and join BK, KC, EL, LF.

## DEMONSTRATION

Because the circles ABC, DEF, are equal, the straight lines from their centres are equal; (III. def. 1)

therefore  $BK, KC$ , are equal to  $EL, LF$ , each to each;  
and the base  $BC$  is equal to the base  $EF$ ; (hyp.)

therefore the angle  $BKC$  is equal to the angle  $ELF$ ; (I. 8)

but equal angles stand upon equal circumferences, when they are at the centres; (III. 26)

therefore the circumference  $BGC$  is equal to the circumference  $EHF$ ;

but the whole circle  $ABC$  is equal to the whole  $EDF$ ; (hyp.)

therefore the remaining parts of the circumferences are equal,

that is,  $BAC$  is equal to  $EDF$ .

Therefore, in equal circles, &c.

Q. E. D.

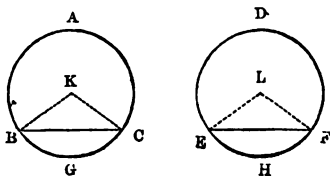
### PROP. XXIX.—THEOREM.

*In equal circles equal circumferences are subtended by equal straight lines.*

(References—Prop. I. 4; III. 1, 27.)

Let  $ABC, DEF$ , be equal circles, and let the circumferences  $BGC, EHF$ , also be equal; and join  $BC, EF$ .

**Then the straight line  $BC$  shall be equal to the straight line  $EF$ .**



### CONSTRUCTION

Take  $K, L$ , the centres of the circles, (III. 1) and join  $BK, KC, EL, LF$ .

### DEMONSTRATION

Because the circumference  $BGC$  is equal to the circumference  $EHF$ ,

therefore the angle  $BKC$  is equal to the angle  $ELF$ ; (III. 27)

and because the circles  $ABC, DEF$ , are equal, the straight lines from their centres are equal; (III. def. 1)

*therefore BK, KC, are equal to EL, LF, each to each ;  
and they contain equal angles ;*

**therefore the base BC is equal to the base EF. (I. 4.)**

Therefore, in equal circles, &c.

Q. E. D.

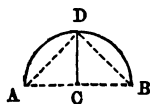
### PROP. XXX.—PROBLEM.

*To bisect a given circumference, that is, to divide it into two equal parts.*

(References—Prop. I. 4, 10, 11 ; III. 1 cor., 28.)

Let ADB be the given circumference.

It is required to bisect it.



#### CONSTRUCTION

Join AB, and bisect it in C ; (I. 10)

from the point C draw CD at right angles to AB. (I. 11.)

**Then the circumference ADB shall be bisected in the point D.**

Join AD, DB.

#### DEMONSTRATION

In the triangles ACD, BCD,

because AC is equal to CB, and CD common to both,

the two sides AC, CD, are equal to the two BC, CD, each to each ;

and the angle ACD is equal to the angle BCD, because each of them is a right angle ;

*therefore the base AD is equal to the base BD ; (I. 4)*

but equal straight lines cut off equal circumferences, (III. 28) the greater equal to the greater, and the less to the less ;

and AD, DB, are each of them less than a semicircle, because DC passes through the centre ; (III. 1 cor.)

*therefore the circumference AD is equal to the circumference DB ;*

**wherefore the circumference ADB is bisected in D.**

Q. E. F.

## PROP. XXXI.—THEOREM.

*In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.*

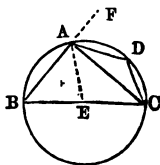
(References—Prop. I. 5, 17, 32; III. 22.)

Let ABCD be a circle, of which the diameter is BC, and the centre E; and draw CA, dividing the circle into the segments ABC, ADC, and join BA, AD, DC.

**Then the angle in the semicircle BAC shall be a right angle;**

**and the angle in the segment ABC, which is greater than a semicircle, shall be less than a right angle;**

**and the angle in the segment ADC, which is less than a semicircle, shall be greater than a right angle.**



## CONSTRUCTION

Join AE, and produce BA to F.

## DEMONSTRATION

Because BE is equal to EA, (I. def. 15)

*the angle EAB is equal to EBA; (I. 5)*

also, because AE is equal to EC,

*the angle EAC is equal to the angle ECA;*

*wherefore the whole angle BAC is equal to the two angles ABC, ACB;*

but FAC, the exterior angle of the triangle ABC, is equal to the two angles ABC, ACB; (I. 32)

*therefore the angle BAC is equal to the angle FAC; (ax. 1)*

*and each of them is therefore a right angle; (I. def. 10)*

**wherefore the angle BAC in a semicircle is a right angle.**

And because the two angles ABC, BAC, of the triangle ABC, are together less than two right angles, (I. 17)

and that  $BAC$  has been proved to be a right angle ;

*therefore  $ABC$  is less than a right angle ;*

**and therefore the angle in a segment  $ABC$ , greater than a semicircle, is less than a right angle.**

And because  $ABCD$  is a quadrilateral figure in a circle, any two of its opposite angles are equal to two right angles ; (III. 22)

*therefore the angles  $ABC$ ,  $ADC$ , are equal to two right angles ;*

and  $ABC$  has been proved to be less than a right angle ;

**wherefore the other  $ADC$  is greater than a right angle.**

Besides, it is manifest that the circumference of the greater segment  $ABC$  falls without the right angle  $CAB$ , but the circumference of the less segment  $ADC$  falls within the right angle  $CAF$ .

‘And this is all that is meant, when in the Greek text, and the translations from it, the angle of the greater segment is said to be greater, and the angle of the less segment is said to be less than a right angle.’

Cor.—From this it is manifest that if one angle of a triangle be equal to the other two it is a right angle, because the angle adjacent to it is equal to the same two ; (I. 32) and when the adjacent angles are equal, they are right angles. (I. def. 10.)

#### PROP. XXXII. — THEOREM.

*If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle ;*

*then the angles made by this line with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.*

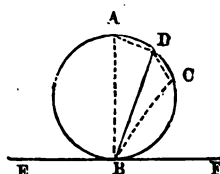
(References — Prop. I. 11, 13, 32 ; III. 19, 22, 31.)

Let the straight line  $EF$  touch the circle  $ABCD$  in  $B$ , and from the point  $B$  let the straight line  $BD$  be drawn cutting the circle.

Then the angles which  $BD$  makes with the touching line  $EF$  shall be equal to the angles in the alternate segments of the circle ;

**that is, the angle  $FBD$  shall be equal to the angle which is in the segment  $DAB$ ,**

**and the angle  $DBE$  to the angle in the segment  $BDC$ .**



## CONSTRUCTION

From the point B draw BA at right angles to EF ; (I. 11)  
and take any point C in the circumference BD, and join AD, DC, CB.

## DEMONSTRATION

Because the straight line EF touches the circle ABCD in the point B,  
and BA is drawn at right angles to the touching line from the point of  
contact B,

the centre of the circle ABCD is in BA ; (III. 19)

therefore the angle ADB in a semicircle, is a right angle, (III. 31)

and consequently the other two angles BAD, ABD, are equal to a right  
angle ; (I. 32)

but ABF is likewise a right angle ; (constr.)

therefore the angle ABF is equal to the angles BAD, ABD ; (ax. 1)

take from these equals the common angle ABD ;

**therefore the remaining angle DBF is equal to the angle  
BAD, which is in the alternate segment of the circle.**  
(ax. 3.)

And because ABCD is a quadrilateral figure in a circle,

the opposite angles BAD, BCD, are equal to two right angles ; (III.  
22)

but the angles DBF, DBE, are likewise equal to two right angles ;  
(I. 13)

therefore the angles DBF, DBE, are equal to the angles BAD, BCD ;

and DBF has been proved equal to BAD ;

**therefore the remaining angle DBE is equal to the angle  
BCD, which is in the alternate segment of the circle.**  
(ax. 3.)

Wherefore, if a straight line, &c.

Q. E. D.



## PROP. XXXIII.—PROBLEM.

*Upon a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.*

(References—Prop. I. 4, 10, 11, 23, 31; III. 16 cor., 31, 32.)

Let AB be the given straight line, and the angle C the given rectilineal angle.

It is required to describe upon the given straight line AB, a segment of a circle, containing an angle equal to the angle C.

*First*, let the angle C be a right angle.



## CONSTRUCTION

Bisect AB in F, (I. 10) and from the centre F, at the distance FB, describe the semicircle AHB.

**Then AHB shall be the segment required.**

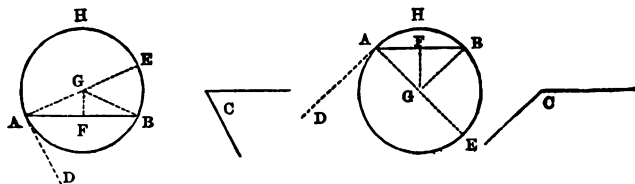
## DEMONSTRATION

Because AHB is a semicircle,

*therefore the angle AHB in it is a right angle, and therefore equal to C; (III. 31)*

**wherefore the angle in the segment AHB is equal to the given angle C.**

*Secondly*, let C be not a right angle.



## CONSTRUCTION

At the point A in the straight line AB, make the angle BAD equal to the angle C, (I. 23)

and from the point A draw AE at right angles to AD; (I. 11)

bisect AB in F, (I. 10) and from F draw FG at right angles to AB, (I. 11) and join GB;

from the centre G, at the distance GA, describe the circle AHB.

**Then AHB is the segment required.**

## DEMONSTRATION

In the triangles AFG, BFG,  
 because AF is equal to FB, and FG common to both,  
 the two sides AF, FG, are equal to the two BF, FG, each to each;  
 and the angle AFG is equal to the angle BFG; (I. def. 10)  
*therefore the base AG is equal to the base GB; (I. 4)*  
*and the circle AHB shall pass through the point B;*  
 and because from the point A, the extremity of the diameter AE, AD  
 is drawn at right angles to AE,  
*therefore AD touches the circle AHB; (III. 16, cor.)*  
 and because AB drawn from the point of contact A cuts the circle,  
*the angle DAB is equal to the angle in the alternate segment AHB;*  
 (III. 32)  
 but the angle DAB is equal to the angle C; (constr.)  
**therefore the angle C is equal to the angle in the segment**  
**AHB.**  
 Wherefore, upon the given straight line AB, the segment AHB of a  
 circle is described which contains an angle equal to the given angle  
 C. Q. E. F.

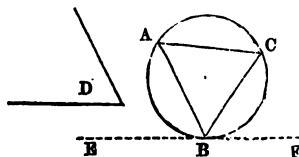
## PROP. XXXIV.—PROBLEM.

*To cut off a segment from a given circle which shall contain an angle equal to a given rectilineal angle.*

(References — Prop. I. 23; III. 17, 32.)

Let ABC be the given circle, and D the given rectilineal angle.

It is required to cut off a segment from the circle ABC that shall contain an angle equal to the angle D.



## CONSTRUCTION

Draw the straight line EF touching the circle ABC in the point B;  
 (III. 17)

and at the point B, in the straight line BF make the angle FBC equal to the angle D. (I. 23.)

**Then the segment BAC shall contain an angle equal to the given angle D.**

#### DEMONSTRATION

Because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B,

*therefore the angle FBC is equal to the angle in the alternate segment BAC of the circle; (III. 32)*

but the angle FBC is equal to the angle D; (constr.)

**therefore the angle in the segment BAC is equal to the angle D. (ax. 1.)**

Wherefore the segment BAC is cut off from the given circle ABC containing an angle equal to the given angle D. Q. E. F.

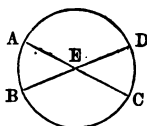
#### PROP. XXXV.—THEOREM.

*If two straight lines within a circle cut one another;  
then the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.*

(References — Prop. I. 12, 47; II. 5; III. 1, 3.)

Let the two straight lines AC, BD, within the circle ABCD, cut one another in the point E.

**Then the rectangle contained by AE, EC, shall be equal to the rectangle contained by BE, ED.**



*First, let AC, BD, pass each of them through the centre.*

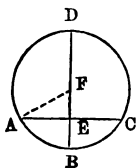
#### DEMONSTRATION

Then because E is the centre,

*therefore AE, EC, BE, ED, are all equal to one another, (I. def. 15)*

**and therefore the rectangle AE, EC, is equal to the rectangle BE, ED.**

*Secondly*, let one of them BD pass through the centre, and cut the other AC, which does not pass through the centre, at right angles in the point E.



## CONSTRUCTION

Then if BD be bisected in F, F is the centre of the circle ABCD. Join AF.

## DEMONSTRATION

Because BD, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles in E, therefore AE, EC, are equal to one another, (III. 3)

and because the straight line BD is cut into two equal parts in the point F, and into unequal parts in the point E,

the rectangle BE, ED, together with the square of EF, is equal to the square of FB; (II. 5) that is to the square of FA;

but the squares of AE, EF, are equal to the square of FA; (I. 47)

therefore the rectangle BE, ED, together with the square of EF, is equal to the squares of AE, EF; (ax. 1)

take away the common square of EF,

and the remainders are equal, that is,

the rectangle BE, ED, is equal to the square of AE, that is to the rectangle AE, EC.

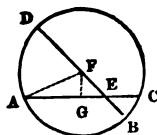
*Thirdly*, let BD, which passes through the centre, cut the other AC, which does not pass through the centre, in E, but not at right angles.

## CONSTRUCTION

Then, as before, if BD be bisected in F, F is the centre of the circle. Join AF, and from F draw FG perpendicular to AC. (I. 12.)

## DEMONSTRATION

Then AG is equal to GC; (III. 3)



*therefore the rectangle AE, EC, together with the square of EG, is equal to the square of AG; (II. 5)*

to each of these equals add the square of GF;

*therefore the rectangle AE, EC, together with the squares of EG, GF, is equal to the squares of AG, GF; (ax. 2)*

but the squares of EG, GF, are equal to the square of EF; (I. 47)

and the squares of AG, GF, are equal to the square of AF;

*therefore the rectangle AE, EC, together with the square of EF, is equal to the square of AF; that is, to the square of FB;*

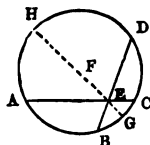
but the square of FB is equal to the rectangle BE, ED, together with the square of EF; (II. 5)

*therefore the rectangle AE, EC, together with the square of EF, is equal to the rectangle BE, ED, together with the square of EF; (ax. 1)*

take away the common square of EF,

**and therefore the remaining rectangle AE, EC, is equal to the remaining rectangle BE, ED. (ax. 3.)**

*Lastly, let neither of the straight lines AC, BD, pass through the centre.*



#### CONSTRUCTION

Take the centre F, (III. 1) and through E, the intersection of the straight lines AC, DB, draw the diameter GEFH.

#### DEMONSTRATION

Because the rectangle AE, EC, is equal, as has been shown, to the rectangle GE, EH;

and for the same reason, the rectangle BE, ED, is equal to the same rectangle GE, EH;

therefore the rectangle  $AE, EC$ , is equal to the rectangle  $BE, ED$ . (ax. 1.)

Wherefore, if two straight lines, &c.

Q. E. D.

### PROP. XXXVI.—THEOREM.

*If from any point without a circle, two straight lines be drawn, one of which cuts the circle, and the other touches it;*

*then the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.*

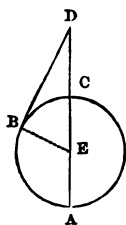
(References — Prop. I. 12, 47; II. 6; III. 1, 3, 18.)

Let  $D$  be any point without the circle  $ABC$ ; and let  $DCA, DB$ , be two straight lines drawn from it, of which  $DCA$  cuts the circle, and  $DB$  touches the same.

**Then the rectangle  $AD, DC$ , shall be equal to the square of  $BD$ .**

Either  $DCA$  passes through the centre, or it does not.

*First*, let it pass through the centre  $E$ .



#### CONSTRUCTION

From the point  $E$  to  $B$ , draw the straight line  $EB$ .

#### DEMONSTRATION

Then the angle  $EBD$  is a right angle; (III. 18)  
and because the straight line  $AC$  is bisected in  $E$ , and produced to the point  $D$ ,

*the rectangle  $AD, DC$ , together with the square of  $EC$ , is equal to the square of  $ED$ ; (II. 6)*

but  $CE$  is equal to  $EB$ ;

*therefore the rectangle AD, DC, together with the square of EB, is equal to the square of ED;*

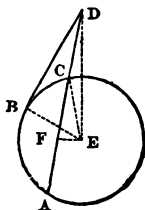
but the square of ED is equal to the squares of EB, BD, because EBD is a right angle; (I. 47)

*therefore the rectangle AD, DC, together with the square of EB, is equal to the squares of EB, BD; (ax. 1)*

take away the common square of EB;

**therefore the remaining rectangle AD, DC, is equal to the square of the tangent DB. (ax. 3.)**

*Secondly, let DCA not pass through the centre of the circle ABC.*



#### CONSTRUCTION

Take the centre E, (III. 1) and draw EF perpendicular to AC, (I. 12) and join EB, EC, ED.

#### DEMONSTRATION

Because the straight line EF, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles, it shall likewise bisect it; (III. 3)

*therefore AF is equal to FC;*

and because the straight line AC is bisected in F, and produced to D, *the rectangle AD, DC, together with the square of FC, is equal to the square of FD; (II. 6)*

to each of these equals add the square of FE;

*therefore the rectangle AD, DC, together with the squares of CF, FE, is equal to the squares of DF, FE; (ax. 2)*

but the square of ED is equal to the squares of DF, FE, because EFD is a right angle; (I. 47)

and for the same reason the square of EC is equal to the squares of CF, FE;

*therefore the rectangle AD, DC, together with the square of EC, is equal to the square of ED; (ax. 1)*

but CE is equal to EB;

*therefore the rectangle AD, DC, together with the square of EB, is equal to the square of ED,*

but the squares of EB, BD, are equal to the square of ED, because EBD is a right angle; (I. 47)

*therefore the rectangle AD, DC, together with the square of EB, is equal to the squares of EB, BD;*

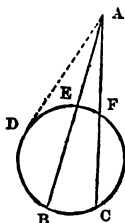
take away the common square of EB;

**therefore the remaining rectangle AD, DC, is equal to the square of DB.** (ax. 3.)

Wherefore, if from any point, &c.

Q. E. D.

Cor. — If from any point without a circle, there be drawn two straight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another.



Thus the rectangle BA, AE, is equal to the rectangle CA, AF, for each of them is equal to the square of the straight line AD, which touches the circle.

#### PROP. XXXVII.—THEOREM.

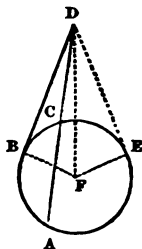
*If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle be equal to the square of the line which meets it; then that line shall touch the circle.*

(References — Prop. I. 8; III. 1, 16 cor., 17, 18, 36.)

Let any point D be taken without the circle ABC, and from it let two straight lines, DCA and DB, be drawn, of which DCA cuts the circle, and DB meets it.



If the rectangle AD, DC, be equal to the square of DB,  
Then DE shall touch the circle.



## CONSTRUCTION

Draw the straight line DE, touching the circle ABC in E; (III. 17)  
find F, the centre of the circle, (III. 1) and join FE, FB, FD.

## DEMONSTRATION

Then FED is a right angle; (III. 18)  
and because DE touches the circle ABC, and DCA cuts it;  
*the rectangle AD, DC, is equal to the square of DE; (III. 36)*  
but the rectangle AD, DC, is, by hypothesis, equal to the square of DB;  
therefore the square of DE is equal to the square of DB; (ax. 1)  
*and the straight line DE equal to the straight line DB;*  
and FE is equal to FB. (I. def. 15.)

Hence in the two triangles DEF, DBF,  
because DE is equal to DB, and EF to FB,  
the two sides DE, EF, are equal to the two DB, BF, each to each;  
and the base FD is common to the two triangles;  
*therefore the angle DEF is equal to the angle DBF; (I. 8)*  
but DEF was shown to be a right angle,  
*therefore also DBF is a right angle; (ax. 1)*  
and FB, if produced, is a diameter,  
and the 'straight line which is drawn at right angles to a diameter, from  
the extremity of it, touches the circle;' (III. 16 cor.)

**therefore DE touches the circle ABC.**

Wherefore, if from a point, &c.

Q. E. D.

# BOOK IV.

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## DEFINITIONS.

### I.

A RECTILINEAL figure is said to be inscribed in another rectilinear figure, when all the angular points of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

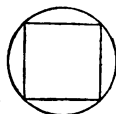


### II.

In like manner a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

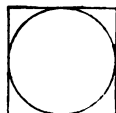
### III.

A rectilinear figure is said to be inscribed in a circle, when all the angular points of the inscribed figure are upon the circumference of the circle.



### IV.

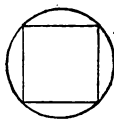
A rectilinear figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.



### V.

In like manner, a circle is said to be inscribed in a rectilinear figure, when the circumference of the circle touches each side of the figure.

## VI.



A circle is said to be described about a rectilinear figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.

## VII.

A straight line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

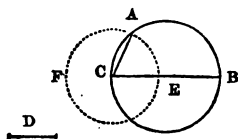
## PROP. I.—PROBLEM.

*In a given circle to place a straight line equal to a given straight line which is not greater than the diameter of the circle.*

(References — Prop. I. 3.)

Let ABC be the given circle, and D the given straight line not greater than the diameter of the circle.

It is required to place in the circle ABC a straight line equal to D.



## CONSTRUCTION

Draw BC the diameter of the circle ABC ;  
then, if BC is equal to D, the thing required is done ;  
for in the circle ABC a straight line BC is placed equal to D.

But, if it is not, BC is greater than D ; (hyp.)

make CE equal to D, (I. 3)

and from the centre C, at the distance CE, describe the circle AEF  
and join CA.

**Then CA shall be equal to D.**

## DEMONSTRATION

Because C is the centre of the circle AEF,

therefore CA is equal to CE ; (I. def. 15)

but D is equal to CE ; (constr.)

therefore **D** is equal to **CA**; (ax. 1)

Wherefore, in the circle **ABC**, a straight line **CA** is placed equal to the given straight line **D**, which is not greater than the diameter of the circle.  
Q. E. F.

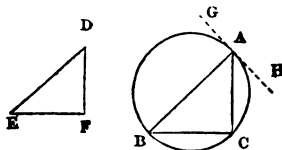
### PROP. II.—PROBLEM.

*In a given circle to inscribe a triangle equiangular to a given triangle.*

(References — Prop. I. 23, 32; III. 17, 32.)

Let **ABC** be the given circle, and **DEF** the given triangle.

It is required to inscribe in the circle **ABC** a triangle equiangular to the triangle **DEF**.



### CONSTRUCTION

Draw the straight line **GAH** touching the circle in the point **A**; (III. 17)

and at the point **A**, in the straight line **AH**, make the angle **HAC** equal to the angle **DEF**; (I. 23)

and at the point **A**, in the straight line **AG**, make the angle **GAB** equal to the angle **DFE**, and join **BC**.

**Then ABC shall be the triangle required.**

### DEMONSTRATION

Because **HAG** touches the circle **ABC**, and **AC** is drawn from the point of contact,

*the angle HAC is equal to the angle ABC in the alternate segment of the circle;* (III. 32)

but **HAC** is equal to the angle **DEF**; (constr.)

*therefore also the angle ABC is equal to DEF;* (ax. 1)

for the same reason, the angle **ACB** is equal to the angle **DFE**;

*therefore the remaining angle BAC is equal to the remaining angle EDF;*  
(I. 32, and ax. 1)

**wherefore the triangle ABC is equiangular to the triangle DEF,**

**and it is inscribed in the circle ABC.**

Q. E. F.

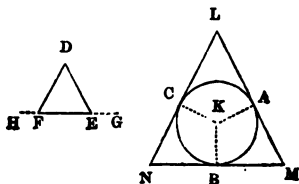
## PROP. III.—PROBLEM.

*About a given circle to describe a triangle equiangular to a given triangle.*

(References—Prop. I. 13, 23, 32; III. 1, 17, 18.)

Let ABC be the given circle, and DEF the given triangle.

It is required to describe a triangle about the circle ABC equiangular to the triangle DEF.



## CONSTRUCTION

Produce EF both ways to the points G, H;  
find the centre K of the circle ABC, (III. 1) and from it draw any straight line KB;  
at the point K, in the straight line KB, make the angle BKA equal to the angle DEG, (I. 23)  
and the angle BKC equal to the angle DFH;  
and through the points A, B, C, draw the straight lines LAM, MBN, NCL, touching the circle ABC. (III. 17.)

**Then LMN shall be the triangle required.**

## DEMONSTRATION

Because LM, MN, NL, touch the circle ABC in the points A, B, C, to which from the centre are drawn KA, KB, KC,  
*the angles at the points A, B, C, are right angles; (III. 18)*  
and because the four angles of the quadrilateral figure AMBK are equal to four right angles, for it can be divided into two triangles;  
and that two of them, KAM, KBM, are right angles,  
*the other two AKB, AMB, are equal to two right angles; (ax. 3)*  
but the angles DEG, DEF, are likewise equal to two right angles, (I. 13)  
*therefore the angles AKB, AMB, are equal to the angles DEG, DEF, (ax. 1)*  
of which AKB is equal to DEG; (constr.)  
*wherefore the remaining angle AMB, is equal to the remaining angle DEF. (ax. 3.)*

In like manner, it may be demonstrated that the angle LNM is equal to DFE ;

and therefore the remaining angle MLN is equal to the remaining angle EDF ; (I. 32, and ax. 3)

wherefore the triangle LMN is equiangular to the triangle DEF ;

and it is described about the circle ABC. Q. E. F.

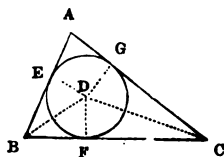
#### PROP. IV.—PROBLEM.

*To inscribe a circle in a given triangle.*

(References—Prop. I. 9, 12, 26 ; III. 16.)

Let the given triangle be ABC.

It is required to inscribe a circle in ABC.



#### CONSTRUCTION

Bisect the angles ABC, BCA, by the straight lines BD, CD, meeting one another in the point D, (I. 9)

from which draw DE, DF, DG, perpendiculars to AB, BC, CA ; (I. 12)  
from the centre D, with the distance DE, describe the circle EFG.

**Then the circle EFG shall be inscribed in the triangle ABC,**

#### DEMONSTRATION

Because the angle EBD is equal to the angle FBD, for the angle ABC is bisected by BD,

and that the right angle BED is equal to the right angle BFD ; (ax. 11)

then, in the triangles EBD, FBD,

the two angles of the one are equal to two angles of the other, each to each ;

and the side BD, which is opposite to one of the equal angles in each, is common to both ;

therefore their other sides are equal; (I. 26)

wherefore  $DE$  is equal to  $DF$ ;

for the same reason,  $DG$  is equal to  $DF$ ;

therefore  $DE$  is equal to  $DG$ ; (ax. 1)

wherefore the three straight lines  $DE$ ,  $DF$ ,  $DG$ , are equal to one another;

and the circle described from the centre  $D$ , at the distance of one of them,  $DE$ , will pass through the extremities of the other two.

And because the angles at the points  $E$ ,  $F$ ,  $G$ , are right angles, and the straight line which is drawn from the extremity of a diameter at right angles to it, touches the circle; (III. 16)

therefore the straight lines  $AB$ ,  $BC$ ,  $CA$ , do each of them touch the circle;

and therefore the circle  $EFG$  is inscribed in the triangle  $ABC$ .  
Q. E. F.

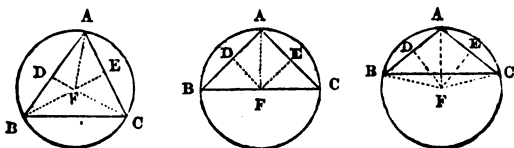
#### PROP. V.—PROBLEM.

To describe a circle about a given triangle.

(References — Prop. I. 4, 10, 11; III. 31.)

Let the given triangle be  $ABC$ .

It is required to describe a circle about  $ABC$ .



#### CONSTRUCTION

Bisect  $AB$ ,  $AC$ , in the points  $D$ ,  $E$ , (I. 10)

and from these points draw  $DF$ ,  $EF$ , at right angles to  $AB$ ,  $AC$ .  
(I. 11.)

$DF$ ,  $EF$ , produced meet one another;

for, if they do not meet, they are parallel,

wherefore  $AB$ ,  $AC$ , which are at right angles to them, are parallel;

which is absurd;

let them meet in  $F$ , and join  $FA$ ;

also if the point  $F$  be not in  $BC$ , join  $BF$ ,  $CF$ ;  
and from the centre  $F$ , at the distance of  $FA$ , describe the circle  
 $ABC$ .

**Then the circle  $ABC$  shall be described about the triangle  
 $ABC$ .**

#### DEMONSTRATION

Because  $AD$  is equal to  $DB$ , and  $DF$  common, and at right angles to  
 $AB$ ,

*therefore the base  $AF$  is equal to the base  $FB$ . (I. 4.)*

In like manner, it may be shown that  $CF$  is equal to  $FA$ ;  
therefore  $BF$  is equal to  $FC$ ; (ax. 1)

*wherefore  $FA$ ,  $FB$ ,  $FC$ , are equal to one another,  
and the circle described from the centre  $F$ , at the distance of one of  
them,  $FA$ , will pass through the extremities of the other two.*

**Therefore the circle  $ABC$  is described about the triangle  
 $ABC$ . Q. E. F.**

Cor. And it is manifest that, when the centre of the circle falls within  
the triangle, *each of its angles is less than a right angle*, each of them  
being in a segment greater than a semicircle; (III. 31)

but when the centre is in one of the sides of the triangle, *the angle  
opposite to this side is a right angle*, being in a semicircle;

and if the centre fall without the triangle, *the angle opposite to the  
side beyond which it is, is greater than a right angle*, being in a segment  
less than a semicircle.

Wherefore, conversely, if the given triangle be acute-angled, *the  
centre of the circle falls within it*;

if it be a right-angled triangle, *the centre is in the side opposite to the  
right angle*;

and, if it be an obtuse-angled triangle, *the centre falls without the  
triangle, beyond the side opposite to the obtuse angle*.

#### PROP. VI. — PROBLEM.

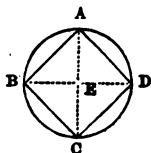
*To inscribe a square in a given circle.*

(References — Prop. I. 4, 11; III. I. 31.)

Let  $ABCD$  be the given circle.

It is required to inscribe a square in  $ABCD$ .





## CONSTRUCTION

Draw the diameters AC, BD, at right angles to one another; (III. 1, and I. 11)

and join AB, BC, CD, DA.

**Then the figure ABCD shall be inscribed in the circle ABCD.**

## DEMONSTRATION

Because BE is equal to ED, for E is the centre, and that EA is common; and at right angles to BD;

*the base BA is equal to the base AD; (I. 4)*

and for the same reason, BC, CD, are each of them equal to BA or AD;

*therefore the quadrilateral figure ABCD is equilateral.*

It is also rectangular;

for the straight line BD, being the diameter of the circle ABCD, BAD is a semicircle;

*wherefore the angle BAD is a right angle; (III. 31)*

for the same reason, each of the angles ABC, BCD, CDA, is a right angle;

*therefore the quadrilateral figure ABCD is rectangular,*

and it has been shown to be equilateral;

*therefore the quadrilateral figure ABCD is a square; (I. def. 30)*

and it touches the circumferences of the circle at each of its angles.

**Therefore the square ABCD is inscribed in the circle ABCD.**

Q. E. F.

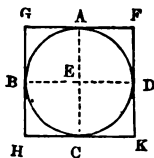
## PROP. VII.—PROBLEM.

*To describe a square about a given circle.*

(References—Prop. I. 28, 34; III. 17, 18.)

Let ABCD be the given circle.

It is required to describe a square about it.



## CONSTRUCTION

Draw two diameters, AC, BD, of the circle ABCD at right angles to one another;  
and through the points A, B, C, D, draw FG, GH, HK, KF, touching the circle; (III. 17)

**Then the figure GHKF, shall be described about the circle ABCD.**

## DEMONSTRATION

Because FG touches the circle ABCD, and EA is drawn from the centre E to the point of contact A,

*the angles at A are right angles; (III. 18)*

for the same reason,

*the angles at the points B, C, D, are right angles;*

and because the angle AEB is a right angle, as likewise is EBG,

*therefore GH is parallel to AC; (I. 28)*

for the same reason,

*AC is parallel to FK;*

and in like manner it may be demonstrated that GF, HK, are each of them parallel to BED;

*therefore the figures GK, GC, AK, FB, BK, are parallelograms;*

*and therefore GF is equal to HK, and GH to FK; (I. 34)*

and because AC is equal to BD, and that AC is equal to each of the two GH, FK;

and BD to each of the two GF, HK;

*therefore GH, FK, are each of them equal to GF, or HK;*

*wherefore the quadrilateral figure GHKF is equilateral.*

It is also rectangular;

for GBEA being a parallelogram, and AEB a right angle,

*therefore AGB is likewise a right angle; (I. 34)*

in the same manner, it may be shown that the angles at H, K, F, are right angles;

therefore the quadrilateral figure GHKF is rectangular ;  
 and it was demonstrated to be equilateral ;  
 therefore GHKF is a square ;  
 and all its sides touch the circumference of the circle ;  
 wherefore the square GHKF is described about the  
 circle ABCD. Q. E. F.

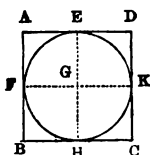
### PROP. VIII.—PROBLEM.

*To inscribe a circle in a given square.*

(References—Prop. I. 10, 29, 31, 34 ; III 16, cor.)

Let ABCD be the given square.

It is required to inscribe a circle in ABCD.



#### CONSTRUCTION

Bisect each of the sides AB, AD, in the points F, E ; (I. 10)  
 and through E draw EH parallel to AB or DC ; (I. 31)  
 and through F draw FK parallel to AD or BC, cutting EH in G ;  
 from the centre G, at the distance GE, describe the circle EFHK.

**Then the circle EFHK shall be inscribed in the square ABCD.**

#### DEMONSTRATION

Because each of the figures AK, KB, AH, HD, AG, GC, BG, GD, is a parallelogram (constr.)

their opposite sides are equal ; (I. 34)

and because AD is equal to AB, (I. def. 30) and that AE is the half of AD, and AF the half of AB,

therefore AE is equal to AF ; (ax. 7)

and the sides opposite to these are equal,

wherefore FG is equal to GE ;

in the same manner it may be demonstrated that  $GH, GK$ , are each of them equal to  $FG$  or  $GE$  ;

*therefore the four straight lines  $GE, GF, GH, GK$ , are equal to one another ;*

*and the circle described from the centre  $G$ , at the distance of one of them,  $GE$ , will pass through the extremities of the other three.*

And because the angles at the points  $E, F, H, K$ , are right angles ;  
(I. 29)

and that the straight line which is drawn from the extremity of a diameter, at right angles to it, touches the circle ; (III. 16, Cor.)

*therefore each of the straight lines  $AB, BC, CD, DA$ , touches the circle  $EFHK$ ,*

**wherefore the circle  $EFHK$  is inscribed in the square  $ABCD$ .**  
Q. E. F.

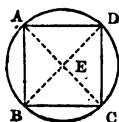
#### PROP. IX. — PROBLEM.

*To describe a circle about a given square.*

(References — Prop. I. 6, 8.)

Let  $ABCD$  be the given square.

It is required to describe a circle about  $ABCD$ .



#### CONSTRUCTION

Join  $AC, BD$ , cutting one another in  $E$  ;

then, from the centre  $E$ , at the distance of  $EA$ , describe the circle  $ABCD$ .

**Then the circle  $ABCD$  shall be described about the square  $ABCD$ .**

#### DEMONSTRATION

Because, in the two triangles,  $DAC, BAC$ ,  $DA$  is equal to  $AB$  (I. def. 30), and  $AC$  common,

the two sides  $DA, AC$ , are equal to the two  $BA, AC$ , each to each ;  
and the base  $DC$  is equal to the base  $BC$  ;

*therefore the angle  $DAC$  is equal to the angle  $BAC$  ; (I. 8)*

*and the angle DAB is bisected by the straight line AC.*

In the same manner it may be demonstrated that the angles ABC, BCD, CDA, are severally bisected by the straight lines BD, AC.

Hence, because the angle DAB is equal to the angle ABC, (I. def. 30) and that the angle EAB is the half of DAB, and EBA the half of ABC;

the angle EAB is equal to the angle EBA; (ax. 7)

therefore the side EA is equal to the side EB. (I. 6.)

In the same manner it may be demonstrated that the straight lines EC, ED, are each of them equal to EA or EB;

therefore the four straight lines EA, EB, EC, ED, are equal to one another;

and the circle described from the centre E, at the distance of one of them, EA, will pass the extremities of the other three;

wherefore the circle ABCD is described about the square ABCD. Q. E. F.

#### PROP. X.—PROBLEM.

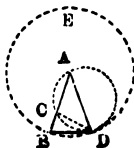
*To describe an isosceles triangle, having each of the angles at the base double of the third angle.*

(References—Prop. I. 5, 6, 32; II. 11; III. 32, 37; IV. 1, 5.)

#### CONSTRUCTION

Take any straight line AB, and divide it in the point C, so that the rectangle AB, BC, may be equal to the square of CA; (II. 11) and from the centre A, at the distance AB, describe the circle BDE, in which place the straight line BD equal to AC, which is not greater than the diameter of the circle BDE; (IV. 1) and join DA.

Then the triangle AED shall be such as is required, that is, each of the angles AED, ADE shall be double of the angle BAD.



Join DC, and about the triangle ADC describe the circle ACD.  
(IV. 5.)

## DEMONSTRATION

Because the rectangle AB, BC, is equal to the square of AC, and that AC is equal to BD, (constr.)

*the rectangle AB, BC, is equal to the square of BD ; (ax. 1)*

and because from the point B without the circle ACD, two straight lines BCA, BD, are drawn to the circumferences, one of which cuts, and the other meets the circle,

and that the rectangle AB, BC, contained by the whole of the cutting line, and the part of it without the circle, is equal to the square of BD, which meets it ;

*the straight line BD touches the circle ACD ; (III. 37)*

and because BD touches the circle, and DC is drawn from the point of contact D,

*the angle BDC is equal to the angle DAC in the alternate segment of the circle ; (III. 32)*

to each of these add the angle CDA ;

*therefore the whole angle BDA is equal to the two angles CDA, DAC ; (ax. 2)*

but the exterior angle BCD is equal to the angles CDA, DAC ;  
(I. 32)

*therefore also BDA is equal to BCD ; (ax. 1)*

but BDA is equal to the angle CBD, (I. 5) because the side AD is equal to the side AB ;

therefore CBD, or DBA, is equal to BCD ; (ax. 1)

*and consequently the three angles BDA, DBA, BCD, are equal to one another.*

Again, because the angle DBC is equal to the angle BCD,

*the side DB is equal to the side DC ; (I. 6)*

but BD was made equal to CA ;

*therefore also CA is equal to CD ; (ax. 1)*

and the angle CDA is equal to the angle DAC ; (I. 5)

*therefore the angles CDA, DAC, together are double of the angle DAC ;*

but BCD is equal to the angles CDA, DAC ; (I. 32)

*therefore also BCD is double of DAC ;*

and BCD was proved to be equal to each of the angles BDA, DBA,

therefore each of the angles  $\angle BDA$ ,  $\angle DRA$ , is double of the angle  $\angle DAB$ .

Wherefore an isosceles triangle  $ABD$  has been described, having each of angles at the base double of the third angle. Q. E. F.

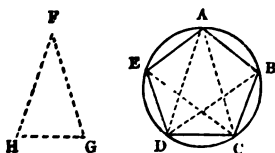
### PROP. XI.—PROBLEM.

*To inscribe an equilateral and equiangular pentagon in a given circle.*

(References — Prop. I. 9 ; III. 26, 27, 29 ; IV. 2, 10.)

Let  $ABCDE$  be the given circle.

It is required to inscribe an equilateral and equiangular pentagon in the circle  $ABCDE$ .



### CONSTRUCTION

Describe an isosceles triangle  $FGH$ , having each of the angles at  $G$ ,  $H$ , double of the angle at  $F$ ; (IV. 10)

and in the circle  $ABCDE$  inscribe the triangle  $ACD$ , equiangular to the triangle  $FGH$ , (IV. 2)

so that the angle  $CAD$  may be equal to the angle at  $F$ , and each of the angles  $ACD$ ,  $CDA$ , equal to the angle at  $G$  or  $H$ ;

wherefore each of the angles  $ACD$ ,  $CDA$ , is double of the angle  $CAD$ ;

bisect the angles  $ACD$ ,  $CDA$ , by the straight lines  $CE$ ,  $DB$ ; (I. 9) and join  $AB$ ,  $BC$ ,  $DE$ ,  $EA$ .

Then  $ABCDE$  shall be the equilateral and equiangular pentagon required.

### DEMONSTRATION

Because each of the angles  $ACD$ ,  $CDA$ , is double of  $CAD$ , and that they are bisected by the straight lines  $CE$ ,  $DB$ ,

the five angles  $\angle DAC$ ,  $\angle ACE$ ,  $\angle ECD$ ,  $\angle CDB$ ,  $\angle BDA$ , are equal to one another;

but equal angles stand upon equal circumferences; (III. 26)

therefore the five circumferences  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$ , are equal to one another;

and equal circumferences are subtended by equal straight lines ;  
(III. 29)

*therefore the five straight lines AB, BC, CD, DE, EA, are equal to one another ;*

**wherefore the pentagon ABCDE is equilateral.**

It is also equiangular ;

for, because the circumference AB is equal to the circumference DE, if to each be added BCD,

*the whole ABCD is equal to the whole EDCB ; (ax. 2)*

but the angle AED stands on the circumference ABCD, and the angle BAE on the circumference EDCB ;

*therefore the angle BAE is equal to the angle AED ; (III. 27)*

for the same reason,

*each of the angles ABC, BCD, CDE, is equal to the angle BAE or AED ;*

**therefore the pentagon ABCDE is equiangular.**

And it has been shown that it is equilateral ;

**wherefore, in the given circle, an equilateral and equiangular pentagon has been inscribed.**  
Q. E. F.

#### PROP. XII.—PROBLEM.

*To describe an equilateral and equiangular pentagon about a given circle.*

(References — Prop. I. 4, 8, 26, 47 ; III. 17, 18, 27 ; IV. 11.)

Let ABCDE be the given circle.

It is required to describe an equilateral and equiangular pentagon about the circle ABCDE.

#### CONSTRUCTION

Let the angular points of a pentagon, inscribed in the circle, by the last proposition, be in the points A, B, C, D, E, so that the circumferences AB, BC, CD, DE, EA, are equal ; (IV. 11)

and through the points A, B, C, D, E, draw GH, HK, KL, LM, MG, touching the circle. (III. 17.)

**Then the figure GHELM shall be the equilateral and equiangular pentagon required.**

Take the centre F, and join FB, FK, FC, FL, FD.

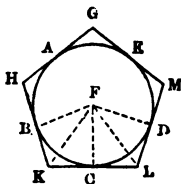
#### DEMONSTRATION

Because the straight line KL touches the circle ABCDE in the point C, to which FC is drawn from the centre F,



**FC is perpendicular to KL ; (III. 18)**

*therefore each of the angles at C is a right angle ;*



for the same reason,

*the angles at the points B, D, are right angles ;*

and because **FCK** is a right angle, .

*the square of FK is equal to the squares of FC, CK ; (I. 47)*

for the same reason, the square of FK is equal to the squares of FB, BK ;

*therefore the squares of FC, CK, are equal to the squares of FB, BK ;*  
(ax. 1)

of which the square of FC is equal to the square of FB ;

*therefore the remaining square of CK is equal to the remaining square of BK ; (ax. 3)*

*and the straight line CK equal to BK.*

Hence in the two triangles BFK, CFK,

because FB is equal to FC, and FK common to both,

the two BF, FK, are equal to the two CF, FK, each to each ;

and the base BK was proved equal to the base KC ;

*therefore the angle BFK is equal to the angle KFC, (I. 8) and the angle BKF to FKC ; (I. 4)*

*wherefore the angle BFC is double of the angle KFC, and BKC double of FKC ;*

for the same reason,

*the angle CFD is double of the angle CFL, and CLD double of CLF ;*

and because the circumference BC is equal to the circumference CD,

*the angle BFC is equal to the angle CFD ; (III. 27)*

and BFC is double of the angle KFC, and CFD double of CFL ;

*therefore the angle KFC is equal to the angle CFL, (ax. 7)*

and the right angle FCK is equal to the right angle FCL.

Hence in the two triangles FKC, FLC,

there are two angles of one equal to two angles of the other, each to each,

and the side FC, which is adjacent to the equal angles in each, is common to both ;

therefore the other sides shall be equal to the other sides, and the third angle to the third angle ; (I. 26)

*therefore the straight line KC is equal to CL, and the angle FKC to the angle FLC ;*

and because KC is equal to CL,

*therefore KL is double of KC ;*

in the same manner it may be shown that

*HK is double of BK.*

And because BK is equal to KC, as was demonstrated, and that KL is double of KC, and HK double of BK,

*therefore HK is equal to KL ; (ax. 6)*

in like manner, it may be shown

*that GH, GM, ML, are each of them equal to HK or KL ;*

**therefore the pentagon GHELM is equilateral.**

It is also equiangular ;

for, since the angle FKC is equal to the angle FLC,

and the angle HKL double of the angle FKC, and KLM double of FLC, as was before demonstrated,

*the angle HKL is equal to KLM ; (ax. 6)*

and in like manner, it may be shown that each of the angles KHG, HGM, GML, is equal to the angle HKL or KLM ;

*therefore the five angles GHK, HKL, KLM, LMG, MGH, are equal to one another,*

**therefore the pentagon GHELM is equiangular.**

and it is equilateral, as was demonstrated ;

and it is described about the circle ABCDE.

Q. E. F.

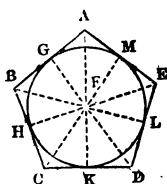
#### PROP. XIII.—PROBLEM.

*To inscribe a circle in a given equilateral and equiangular pentagon.*

(References—Prop. I. 4, 9, 12, 26 ; III. 16.)

Let ABCDE be the given equilateral and equiangular pentagon.

*It is required to inscribe a circle in the pentagon ABCDE.*



## CONSTRUCTION

Bisect the angles BCD, CDE, by the straight lines CF, DF; (I. 9)  
 and from the point F, in which they meet, draw the straight lines  
 FB, FA, FE;  
 draw also the lines FG, FH, FK, FL, and FM, perpendicular to the  
 sides of the pentagon; (I. 12)  
 and from the centre F, at the distance FG, describe the circle  
 GHKLM.

**Then GHKLM shall be inscribed as required in the  
 pentagon ABCDE.**

## DEMONSTRATION

Because in the two triangles BCF, DCF,  
 BC is equal to CD (hyp.), and CF common,  
 the two sides BC, CF, are equal to the two DC, CF, each to each;  
 and the angle BCF is equal to the angle DCF; (constr.)

*therefore the base BF is equal to the base FD, (I. 4)*

and the other angles to the other angles, to which the equal sides are  
 opposite;

*therefore the angle CBF is equal to the angle CDF;*

and because the angle CDE is double of CDF,  
 and that CDE is equal to CBA, and CDF to CBF;

CBA is also double of CBF;

therefore the angle ABF is equal to the angle CBF

*wherefore the angle ABC is bisected by the straight line BF.*

In the same manner it may be demonstrated that

*the angles BAE, AED, are bisected by the straight lines AF, FE.*

And because the angle HCF is equal to KCF, and the right angle  
 FHC equal to the right angle FKC;

then in the two triangles FHC, FKC,

there are two angles of the one equal to two angles of the other, each  
 to each;

and the side FC, which is opposite to one of the equal angles in each,  
*is common to both;*

therefore the other sides shall be equal, each to each ; (I. 26)

*wherefore the perpendicular FH is equal to the perpendicular FK.*

In the same manner it may be demonstrated that FL, FM, FG, are each of them equal to FH or FK ;

therefore the five straight lines FG, FH, FK, FL, FM, are equal to one another ;

*wherefore the circle described from the centre F, at the distance of one of them, FG, will pass through the extremities of the other four.*

And because the angles at the points G, H, K, L, M, are right angles, and that a straight line drawn from the extremity of the diameter of a circle at right angles to it, touches the circle ; (III. 16)

*therefore each of the straight lines AB, BC, CD, DE, EA touches the circle ;*

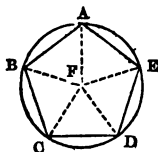
**wherefore the circle GHELM is inscribed in the pentagon ABCDE.** Q. E. F.

#### PROP. XIV.—PROBLEM.

*To describe a circle about a given equilateral and equiangular pentagon.*

(References—Prop. I. 6, 9.)

Let ABCDE be the given equilateral and equiangular pentagon. It is required to describe a circle about it.



#### CONSTRUCTION

Bisect the angles BCD, CDE, by the straight lines FC, FD, (I. 9) and from the point F, in which they meet, draw the straight lines FB, FA, FE, to the points B, A, E ;

from the centre F, at the distance FA, FB, FC, FD, or FE, describe the circle ABCDE.

**Then ABCDE shall be described as required about the given pentagon,**

## DEMONSTRATION

It may be demonstrated, as in the same manner as in the preceding proposition, that

*the angles CBA, BAE, AED, are bisected by the straight lines FB, FA, FE;*

and because the angle BCD is equal to the angle CDE,  
and that FCD is the half of the angle BCD, and CDF the half of CDE;

the angle FCD is equal to FDC; (ax. 7)

*wherefore the side CF is equal to the side FD. (I. 6.)*

In like manner it may be demonstrated that FB, FA, FE, are each of them equal to FC or FD;

therefore the five straight lines FA, FB, FC, FD, FE, are equal to one another;

*and the circle described from the centre F, at the distance of one of them, shall pass through the extremities of the other four,*

**wherefore the circle ABCDE is described about the equilateral and equiangular pentagon ABCDE. Q. E. F.**

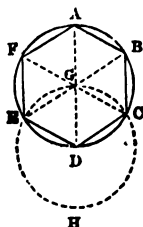
## PROP. XV.—PROBLEM.

*To inscribe an equilateral and equiangular hexagon in a given circle.*

(References — Prop. I. 5 cor., 13, 15, 32; III. 1, 26, 27, 29.)

Let ABCDEF be the given circle.

It is required to inscribe an equilateral and equiangular hexagon in it.



## CONSTRUCTION

Find the centre G of the circle ABCDEF, (III. 1) and draw the diameter AGD;

and from D as a centre, at the distance DG, describe the circle EGCH;

join EG, CG, and produce them to the points B, F; and join AB, BC, CD, DE, EF, FA.

**Then ABCDEF shall be an equilateral and equiangular hexagon inscribed as required in the given circle.**

#### DEMONSTRATION

Because G is the centre of the circle ABCDEF,

*therefore GE is equal to GD;*

and because D is the centre of the circle EGCH,

*therefore DE is equal to DG;*

wherefore GE is equal to ED, (ax. 1)

*and the triangle EGD is equilateral;*

and therefore its three angles EGD, GDE, DEG, are equal to one another; (I. 5 cor.)

but the three angles of a triangle are equal to two right angles; (I. 32)

*therefore the angle EGD is the third part of two right angles;*

in the same manner it may be demonstrated that

*the angle DGC is also the third part of two right angles;*

and because the straight line GC makes with EB the adjacent angles, EGC, CGB, equal to two right angles, (I. 13)

the remaining angle CGB is the third part of two right angles;

*therefore the angles EGD, DGC, CGB, are equal to one another;*

and to these are equal the vertical opposite angles BGA, AGF, FGE; (I. 15)

*therefore the six angles EGD, DGC, CGB, BGA, AGF, FGE, are equal to one another;*

but equal angles stand upon equal circumferences; (III. 26)

*therefore the six circumferences AB, BC, CD, DE, EF, FA, are equal to one another;*

and equal circumferences are subtended by equal straight lines; (III. 29)

*therefore the six straight lines are equal to one another,*

**and the hexagon ABCDEF is equilateral.**

It is also equiangular;

for, since the circumference AF is equal to ED,

to each of these add the circumference ABCD;

*therefore the whole circumference FABCD is equal to the whole EDCBA;*

and the angle FED stands upon the circumference FABCD, and the angle AFE upon EDCBA;

*therefore the angle AFE is equal to FED; (III. 27)*

in the same manner it may be demonstrated that

*the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED;*

**therefore the hexagon ABCDEF is equiangular;**

and it is equilateral, as was shown;

and it is inscribed in the given circle ABCDEF.

Q. E. F.

Cor. From this it is manifest that the side of the hexagon is equal to the straight line from the centre, that is, to the semidiameter of the circle.

And if through the points A, B, C, D, E, F, there be drawn straight lines touching the circle, an equilateral and equiangular hexagon shall be *described about it*, which may be demonstrated from what has been said of the pentagon; and likewise a circle may be *inscribed in* a given equilateral and equiangular hexagon, and *circumscribed about it*, by a method like that used for the pentagon.

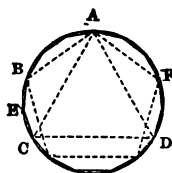
#### PROP. XVI.—PROBLEM.

*To inscribe an equilateral and equiangular quindecagon in a given circle.*

(References—Prop. III. 27, 30; IV. 1, 2, 11.)

Let ABCD be the given circle.

It is required to inscribe an equilateral and equiangular quindecagon in it.



#### CONSTRUCTION

Let AC be the side of an equilateral triangle inscribed in the circle ABCD, (IV. 2)

and AB the side of an equilateral and equiangular pentagon inscribed in the same; (IV. 11)

bisect the circumference BC in E, (III. 30) and join BE, EC ;  
draw straight lines equal to these, and place them contiguous to each other round the circle. (IV. 1.)

**Then ABCDF shall be an equilateral and equiangular quindecagon described as required in the circle ABCD.**

#### DEMONSTRATION

Because of such equal parts as the whole circumference ABCDF contains fifteen, the circumference ABC, which is the third part of the whole, contains five ;

and the circumference AB, which is the fifth part of the whole, contains three ;

*therefore their difference BC contains two such parts,*

but BC was bisected in E ; (constr.) therefore BE, EC are each of them the fifteenth part of the whole circumference ABCD ;

**wherefore the figure ABECDF is equilateral.**

And because each of its angles stands upon thirteen-fifteenths of the circumference,

**therefore the figure ABECDF is equiangular.** (III. 27.)

Wherefore an equilateral and equiangular quindecagon has been inscribed in the circle ABC. Q. E. F.

And in the same manner as was done in the pentagon, if through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon shall be *described about it* ; and likewise, as in the pentagon, a circle may be *inscribed in* a given equilateral and equiangular quindecagon, and *circumscribed about it*.



## BOOK V.

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### DEFINITIONS.

#### I.

A **LESS** magnitude is said to be a *part* of a greater magnitude, when the less measures the greater; that is, when the less is contained a certain number of times exactly in the greater.

#### II.

A greater magnitude is said to be a *multiple* of a less, when the greater is measured by the less; that is, when the greater contains the less a certain number of times exactly.

#### III.

Ratio is a mutual relation of two magnitudes of the same kind to one another, in respect of quantity.

#### IV.

Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

#### V.

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the

fourth ; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth ; or if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

## VI.

Magnitudes which have the same ratio are called proportionals.

When four magnitudes are proportionals, it is usually expressed by saying, the first is to the second, as the third to the fourth.

## VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth ; then, the first is said to have to the second a greater ratio than the third magnitude has to the fourth ; and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

## VIII

Analogy, or proportion, is the similitude of ratios.

## IX.

Proportion consists in three terms at least.

## X.

When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

## XI.

When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, quadruplicate, &c., increasing the denomination still by unity, in any number of proportionals.

Definition A, to wit, of compound ratio.

When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D, the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D ; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F ; and B to C the same ratio that G has to H ; and C to D the same ratio that K has to L ; then, by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F, G to H, and K to L. And the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D ; then, for shortness' sake, M is said to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

## XII.

In proportionals, the antecedent terms are called homologous to one another, as also the consequents to one another.

Geometers make use of the following technical words, to signify certain ways of changing either the order or magnitude of proportionals, so that they continue still to be proportionals.

## XIII.

*Permutanao*, or *alternando*, by permutation or alternately. This word is used when there are four proportionals, and it is inferred that the first has the same ratio to the third which the second has to the fourth ; or that the first is to the third as the second to the fourth : as is shown in Prop. xvi. of this Fifth Book.

## XIV.

*Invertendo*, by inversion ; when there are four proportionals, and it is inferred that the second is to the first as the fourth to the third. —Prop. B. Book v.

## XV.

*Componendo*, by composition ; when there are four proportionals, and it is inferred that the first together with the second, is to the second, as the third together with the fourth, is to the fourth. —Prop. xviii. Book v.

## XVI.

*Dividendo*, by division ; when there are four proportionals, and it is inferred, that the excess of the first above the second, is to the second, as the excess of the third above the fourth, is to the fourth. —Prop. xvii. Book v.

## XVII.

*Convertendo*, by conversion ; when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth. —Prop. E. Book v.

## XVIII.

*Ex æquali* (sc. distantia), or *ex æquo*, from equality of distance ; when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each rank, and it is inferred, that the first is to the last of the first rank of magnitudes, as the first is to the last of the others. Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken, two and two.

## XIX.

*Ex æquali*, from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank ; and as the second is to the third of the first rank, so is the second to the third of the other ; and so on in order ; and the inference is as mentioned in the preceding definition ; whence this is called ordinate proportion. It is demonstrated in Prop. xxii. Book v.

## XX.

*Ex æquali in proportione perturbatâ seu inordinatâ*, from equality in perturbate or disorderly proportion.\* This term is used when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank ; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank ; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank ; and so on in a cross order ; and the inference is as in the 18th Definition. —Prop. xxiii. Book v.

\* Prop. 4. Lib. II Archimedis de sphaerâ et cylindro.

## AXIOMS.

## I.

Equimultiples of the same, or of equal magnitudes, are equal to one another.

## II.

Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

## III.

A multiple of a greater magnitude is greater than the same multiple of a less.

## IV.

That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

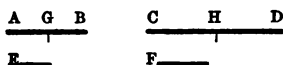
## PROP. I.—THEOREM.

*If any number of magnitudes be equimultiples of as many others, each of each;*

*then what multiple soever any one of them is of its part, the same multiple shall all the first magnitudes be of all the others.*

Let any number of magnitudes AB, CD be equimultiples of as many others E, F, each of each.

**Then whatsoever multiple AB is of E, the same multiple shall AB and CD together be of E and F together.**



## CONSTRUCTION

Because AB is the same multiple of E that CD is of F, as many magnitudes as there are in AB equal to E, so many are there in CD equal to F;

divide AB into magnitudes equal to E, viz. AG, GB; and CD into CH, HD, equal each of them to F.

DEMONSTRATION

Because the number of the magnitudes CH and HD is equal to the number of the others AG and GB ;  
and AG is equal to E, and CH to F ;

*therefore AG and CH together are equal to E and F together ;*

for the same reason, because GB is equal to E, and HD to F ;  
therefore GB and HD together are equal to E and F together ;

*wherefore as many magnitudes as there are in AB equal to E, so many are there in AB, CD together, equal to E and F together ;*

**therefore, whatsoever multiple AB is of E, the same multiple is AB and CD together, of E and F together.**

Therefore, if any magnitudes, how many soever, be equimultiples of as many, each of each ; whatsoever multiple any one of them is of its part, the same multiple shall all the first magnitudes be of all the other : For the same demonstration holds in any number of magnitudes, which was here applied to two. Q. E. D.

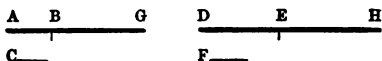
PROP. II.—THEOREM.

*If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth ;*

*then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.*

Let AB the first be the same multiple of C the second, that DE the third is of F the fourth ; and BG the fifth the same multiple of C the second, that EH the sixth is of F the fourth.

**Then shall AG, the first together with the fifth, be the same multiple of C the second, that DH, the third together with the sixth, is of F the fourth.**



DEMONSTRATION

Because AB is the same multiple of C that DE is of F ;

*there are as many magnitudes in AB equal to C, as there are in DE equal to F ;*

in like manner, as many as there are in BG equal to C, so many are there in EH equal to F ;

*therefore as many as there are in the whole AG equal to C, so many are there in the whole DH equal to F ;*

therefore AG is the same multiple of C that DH is of F ;

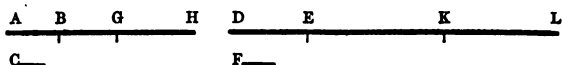
**that is, AG, the first and fifth together, is the same multiple of the second C, that DH, the third and sixth together, is of the fourth F.**

If, therefore, the first be the same multiple, &c.

Q. E. D.

Cor. From this it is plain, that if any number of magnitudes AB, BG, GH be multiples of another C ; and as many DE, EK, KL be the same multiples of F, each of each ;

then the whole of the first, viz. AH, is the same multiple of C, that the whole of the last, viz. DL, is of F.



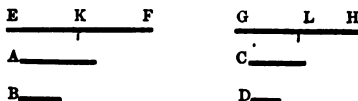
### PROP. III.—THEOREM.

*If the first be the same multiple of the second, which the third is of the fourth ; and if of the first and third there be taken equimultiples ; then these shall be equimultiples, the one of the second, and the other of the fourth.*

(Reference—Prop. v. 2.)

Let A the first, be the same multiple of B the second, that C the third is of D the fourth ; and of A, C, let equimultiples EF, GH be taken.

**Then EF shall be the same multiple of B, that GH is of D.**



### CONSTRUCTION

Divide EF into the magnitudes EK, KF, each equal to A, and GH into GL, LH, each equal to C.

## DEMONSTRATION

Because the number of the magnitudes EK and KF is equal to the number of the others GL and LH ;

and A is the same multiple of B, that C is of D,  
and EK is equal to A, and GL equal to C ; (constr.)

*therefore EK is the same multiple of B, that GL is of D.*

For the same reason, KF is the same multiple of B that LH is of D;  
and so, if there be more parts in EF, GH equal to A, C.

Because, then, the first EK is the same multiple of the second B, which the third GL is of the fourth D ;

and the fifth KF is the same multiple of the second B, which the sixth LH is of the fourth D ;

**therefore EF the first, together with the fifth, is the same multiple of the second B, which GH the third, together with the sixth, is of the fourth D. (v. 2.)**

If, therefore, the first, &c.

Q. E. D.

## PROP. IV.—THEOREM.

*If the first of four magnitudes has the same ratio to the second which the third has to the fourth ;*

*then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz. ' the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.'*

(References—Prop. v. 3, def. 5.)

Let A the first, have to B the second, the same ratio which the third C has to the fourth D ; and of A and C let there be taken any equimultiples whatever E, F ; and of B and D any equimultiples whatever G, H.

**Then E shall have the same ratio to G, which F has to H.**

|         |         |
|---------|---------|
| K _____ | M _____ |
| E _____ | G _____ |
| A _____ | B _____ |
| C _____ | D _____ |
| F _____ | H _____ |
| L _____ | N _____ |



## CONSTRUCTION

Take of *E* and *F* any equimultiples whatever *K*, *L*, and of *G*, *H* any equimultiples whatever *M*, *N*.

## DEMONSTRATION

Then because *E* is the same multiple of *A*, that *F* is of *C* ;  
and of *E* and *F* have been taken equimultiples *K*, *L* ;

*therefore K is the same multiple of A, that L is of C. (v. 3.)*

For the same reason, *M* is the same multiple of *B*, that *N* is of *D*.

And because, as *A* is to *B*, so is *C* to *D* ; (hyp.)

and of *A* and *C* have been taken certain equimultiples *K*, *L* ;

and of *B* and *D* have been taken certain equimultiples *M*, *N* ;

*therefore if K be greater than M, L is greater than N ; and if equal, equal ; and if less, less ; (v. def. 5.)*

but *K*, *L* are any equimultiples whatever of *E*, *F*, and *M*, *N* any whatever of *G*, *H* ; (constr.)

**therefore as *E* is to *G*, so is *F* to *H*. (v. def. 5.)**

Therefore, if the first, &c.

Q. E. D.

Cor. Likewise, if the first has the same ratio to the second, which the third has to the fourth, then also any equimultiples whatever of the first and third shall have the same ratio to the second and fourth. And in like manner, the first and the third shall have the same ratio to any equimultiples whatever of the second and fourth.

Let *A* the first have to *B* the second the same ratio which the third *C* has to the fourth *D* ; and of *A* and *C* let *E* and *F* be any equimultiples whatever.

**Then *E* shall be to *B* as *F* to *D*.**

## CONSTRUCTION

Take of *E*, *F* any equimultiples whatever *K*, *L*, and of *B*, *D* any equimultiples whatever *G*, *H*.

## DEMONSTRATION

Then it may be demonstrated as before, that *K* is the same multiple of *A*, that *L* is of *C* ;

and because *A* is to *B*, as *C* is to *D*, (hyp.)

and of *A* and *C* certain equimultiples have been taken, viz. *K* and *L* ;

and of *B* and *D* certain equimultiples *G*, *H* ;

*therefore, if K be greater than G, L is greater than H ; and if equal, equal ; if less, less ; (v. def. 5.)*

but K, L are any equimultiples whatever of E, F, and G, H any whatever of B, D ; (constr.)

**therefore, as E is to F, so is G to H.** (v. def. 5.)

And in the same way the other case is demonstrated.

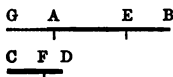
### PROP. V.—THEOREM.

*If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other ;  
then the remainder shall be the same multiple of the remainder, that the whole is of the whole.*

(References—Prop. v. 1, ax. 1.)

Let the magnitude AB be the same multiple of CD, that AE taken from the first, is of CF taken from the other.

**Then the remainder EB shall be the same multiple of the remainder FD, that the whole AB is of the whole CD.**



### CONSTRUCTION .

Take AG the same multiple of FD, that AE is of CF.

### DEMONSTRATION

Then AE is the same multiple of CF, that EG is of CD ; (v. 1.)

but AE is the same multiple of CF, that AB is of CD ; (hyp.)

therefore EG is the same multiple of CD that AB is of CD ;

*wherefore EG is equal to AB ;* (v. ax. 1.)

take from each of them the common magnitude AE ;

*then the remainder AG is equal to the remainder EB.*

Then, because AE is the same multiple of CF, that AG is of FD, (constr.)

and that AG has been proved equal to EB ;

*therefore AE is the same multiple of CF, that EB is of FD ;*

but AE is the same multiple of CF that AB is of CD ; (hyp.)

**therefore EB is the same multiple of FD, that AB is of CD.**

Therefore, if one magnitude, &c.

Q. E. D.

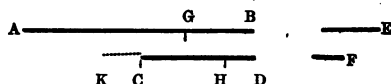
## PROP. VI.—THEOREM.

*If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two ;  
then the remainders are either equal to these others, or equimultiples of them.*

(References—Prop. v. 2, ax. 1.)

Let the two magnitudes AB, CD be equimultiples of the two E, F ;  
and let AG, CH taken from the first two be equimultiples of the same E, F.

**Then the remainders GB, HD, shall be either equal to E, F, or equimultiples of them.**



*First*, let GB be equal to E. Then HD shall be equal to F.

## CONSTRUCTION

Make CK equal to F.

## DEMONSTRATION

Then because AG is the same multiple of E, that CH is of F ; (hyp.)  
and that GB is equal to E, and CK to F ;

*therefore AB is the same multiple of E, that KH is of F ;*

but AB is the same multiple of E, that CD is of F ; (hyp.)

*therefore KH is the same multiple of F, that CD is of F ;*

*wherefore KH is equal to CD ; (v. ax. 1.)*

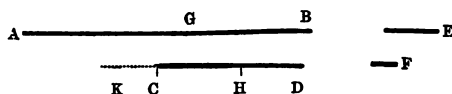
take away the common magnitude CH,

*then the remainder KC is equal to the remainder HD ;*

but KC is equal to F ; (constr.)

**therefore HD is equal to F.**

*Secondly*, let GB be a multiple of E. Then HD shall be the same multiple of F.



## CONSTRUCTION

Make CK the same multiple of F, that GB is of E.

## DEMONSTRATION

Then because AG is the same multiple of E, that CH is of F; (hyp.) and GB the same multiple of E, that CK is of F; (constr.)

*therefore AB is the same multiple of E, that KH is of F; (v. 2.)*

but AB is the same multiple of E, that CD is of F; (hyp.)

*therefore KH is the same multiple of F, that CD is of F;*

*wherefore KH is equal to CD; (v. ax. 1.)*

take away CH from both.

*therefore the remainder KC is equal to the remainder HD;*

and because GB is the same multiple of E, that KC is of F, (constr.) and that KC is equal to HD;

**therefore HD is the same multiple of F, that GB is of E.**

If, therefore, two magnitudes, &c.

Q. E. D.

## PROP A.—THEOREM.

*If the first of four magnitudes has the same ratio to the second which the third has to the fourth;*

*then, if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.*

(Reference—v. def. 5.)

Take any equimultiples of each of them, as the doubles of each.

Then, if the double of the first be greater than the double of the second, *the double of the third is greater than the double of the fourth;* (v. def. 5.)

but, if the first be greater than the second,

*the double of the first is greater than the double of the second;*

*therefore also the double of the third is greater than the double of the fourth;*

*therefore the third is greater than the fourth.*

In like manner, if the first be equal to the second or less than it,

*the third can be proved to be equal to the fourth, or less than it.*

Therefore, if the first, &c.

Q. E. D.

## PROP. B.—THEOREM.

*If four magnitudes are proportionals, they are proportionals also when taken inversely.*

(Reference—v. def. 5.)

Let A be to B, as C is to D.

**Then also inversely, B shall be to A, as D to C.**

|         |         |         |         |
|---------|---------|---------|---------|
| A _____ | B _____ | C _____ | D _____ |
| G _____ | E _____ | H _____ | F _____ |

## CONSTRUCTION

Take of B and D any equimultiples whatever E and F ; and of A and C any equimultiples whatever G and H.

## DEMONSTRATION

Then, first, if E be greater than G,

G is less than E;

and because A is to B, as C is to D, (hyp.)

and of A and C, the first and third, G and H are equimultiples;

and of B and D, the second and fourth, E and F are equimultiples,

and that G is less than E,

*therefore H is less than F ; (v. def. 5.) that is F is greater than H ;*

if, then, E be greater than G,

*therefore F is greater than H.*

In like manner, if E be equal to G, F may be shown to be equal to H;

and if less, less ;

but E, F are any equimultiples whatever of B and D, and G, H any whatever of A and C ; (constr.)

**therefore, as B is to A, so is D to C. (v. def. 5.)**

Therefore, if four magnitudes, &c.

Q. E. D.

PROP. C.—THEOREM.

*If the first be the same multiple of the second, or the same part of it, that the third is of the fourth;*

*then the first is to the second, as the third is to the fourth.*

(References—Prop. v. 3, B; v. def. 5.)

First, let A the first be the same multiple of B the second, that C the third is of D the fourth.

**Then A shall be to B as C is to D.**

A\_\_\_\_\_ B\_\_\_\_\_ C\_\_\_\_\_ D\_\_\_\_\_

E\_\_\_\_\_ G\_\_\_\_\_ F\_\_\_\_\_ H\_\_\_\_\_

CONSTRUCTION

Take of A and C any equimultiples whatever E and F, and of B and D any equimultiples whatever G and H.

DEMONSTRATION

Then, because A is the same multiple of B that C is of D; (hyp.) and that E is the same multiple of A, that F is of C; (constr.)

*therefore E is the same multiple of B, that F is of D; (v. 3.)*

*that is, E and F are equimultiples of B and D;*

but G and H are equimultiples of B and D; (constr.)

*therefore, if E be a greater multiple of B than G is of B, F is a greater multiple of D than H is of D;*

*that is, if E be greater than G, F is greater than H.*

In like manner, if E be equal to G, or less than it, F may be shown to be equal to H, or less than it;

but E, F are equimultiples, any whatever, of A, C, and G, H any equimultiples whatever of B, D; (constr.)

**therefore A is to B, as C is to D. (v. def. 5.)**

Next, let A the first be the same part of B the second, that C the third is of D the fourth.

**Then A shall be to B, as C is to D.**

A\_\_\_\_\_ B\_\_\_\_\_ C\_\_\_\_\_ D\_\_\_\_\_

For since A is the same part of B that C is of D,

therefore B is the same multiple of A that D is of C;

*wherefore, by the preceding case, B is to A, as D is to C;*

**and, therefore, inversely, A is to B, as C is to D. (v. B.)**

Therefore, if the first be the same multiple, &c.

Q. E. D.

## PROP. D.—THEOREM.

*If the first be to the second as the third to the fourth, and if the first be a multiple, or a part of the second;*

*then the third is the same multiple, or the same part of the fourth.*

(References—Prop. v. 4 cor., A, B.)

Let A be to B as C is to D;

and first, let A be a multiple of B.

**Then C shall be the same multiple of D.**

|         |         |         |         |
|---------|---------|---------|---------|
| A _____ | B _____ | C _____ | D _____ |
|         | E _____ |         | F _____ |

## CONSTRUCTION

Take E equal to A, and whatever multiple A or E is of B, make F the same multiple of D.

## DEMONSTRATION

Then, because A is to B, as C is to D; (hyp.)

and of B the second, and D the fourth, equimultiples have been taken E and F;

*therefore A is to E, as C to F; (v. 4 cor.)*

but A is equal to E; (constr.)

*therefore C is equal to F; (v. A.)*

and F is the same multiple of D, that A is of B; (constr.)

**therefore C is the same multiple of D, that A is of B.**

*Next, let A the first be a part of B the second.*

**Then C the third shall be the same part of D the fourth.**

|         |         |         |         |
|---------|---------|---------|---------|
| A _____ | B _____ | C _____ | D _____ |
|---------|---------|---------|---------|

## DEMONSTRATION

Because A is to B, as C is to D; (hyp.)

*then, inversely, B is to A, as D to C; (v. B.)*

but A is a part of B, that is B is a multiple of A; (hyp.)

*therefore, by the preceding case, D is the same multiple of C;*

**that is, C is the same part of D, that A is of B.**

Therefore, if the first, &c.

Q. E. D.

## PROP. VII.—THEOREM.

*Equal magnitudes have the same ratio to the same magnitude ; and the same has the same ratio to equal magnitudes.*

(References—v. def. 5, ax. 1.)

Let A and B be equal magnitudes, and C any other.

**Then A and B shall each of them have the same ratio to C ;**

**and C shall have the same ratio to each of the magnitudes A and B.**

|        |        |        |
|--------|--------|--------|
| A_____ | B_____ | C_____ |
| D_____ | E_____ | F_____ |

## CONSTRUCTION

Take of A and B any equimultiples whatever D and E, and of C any multiple whatever F.

## DEMONSTRATION

Then, because D is the same multiple of A, that E is of B; (constr.) and that A is equal to B; (hyp.)

therefore D is equal to E; (v. ax. 1.)

*therefore, if D be greater than F, E is greater than F; and if equal, equal; if less, less;*

but D, E are any equimultiples of A, B, and F is any multiple of C; (constr.)

**therefore, as A is to C, so is B to C.** (v. def. 5.)

Likewise C shall have the same ratio to A, that it has to B.

For the same construction being made, it may be shown, as before, that D is equal to E ;

*therefore, if F be greater than D, it is likewise greater than E; and if equal, equal; and if less, less;*

but F is any multiple whatever of C, and D, E are any equimultiples whatever of A, B ;

**therefore, C is to A as C is to B.** (v. def. 5.)

Therefore, equal magnitudes, &c.

Q. E. D.



## PROP. VIII.—THEOREM.

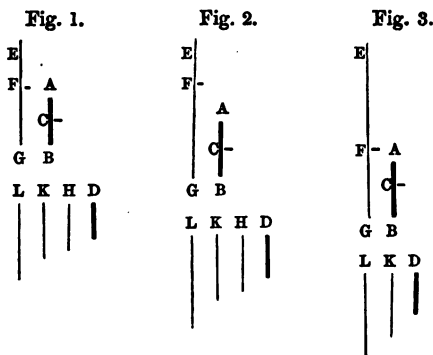
*Of unequal magnitudes, the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less than it has to the greater.*

(References—Prop. v. 1, def. 7.)

Let  $AB$ ,  $BC$ , be two unequal magnitudes, of which  $AB$  is the greater, and let  $D$  be any other magnitude.

**Then  $AB$  shall have a greater ratio to  $D$  than  $BC$  has to  $D$ ;**

**and  $D$  shall have a greater ratio to  $BC$  than it has to  $AB$ .**



## CONSTRUCTION

If the magnitude which is not the greater of the two  $AC$ ,  $CB$ , be not less than  $D$ , take  $EF$ ,  $FG$ , the doubles of  $AC$ ,  $CB$ ; (as in fig. 1.)

but if that which is not the greater of the two  $AC$ ,  $CB$ , be less than  $D$ , (as in fig. 2 and 3.) this magnitude can be multiplied, so as to become greater than  $D$ , whether it be  $AC$  or  $CB$ .

Let it be multiplied until it become greater than  $D$ , and let the other be multiplied as often;

and let  $EF$  be the multiple thus taken of  $AC$ , and  $FG$  the same multiple of  $CB$ ;

therefore  $EF$  and  $FG$  are each of them greater than  $D$ ;

and in all cases, take  $H$  the double of  $D$ ,  $K$  its triple, and so on, till the multiple of  $D$  be that which first becomes greater than  $FG$ ;

let  $L$  be that multiple of  $D$  which is first greater than  $FG$ , and  $K$  the multiple of  $D$  which is next less than  $L$ .

DEMONSTRATION

Then because L is the multiple of D, which is the first that becomes greater than FG,

*the next preceding multiple K is not greater than FG ; that is, FG is not less than K ;*

and since EF is the same multiple of AC, that FG is of CB ; (constr.)

therefore FG is the same multiple of CB, that EG is of AB ; (v. 1.)

*that is, EG and FG are equimultiples of AB and CB.*

And it was shown, that FG is not less than K,

and by construction, EF is greater than D ;

*therefore the whole EG is greater than K and D together ;*

but K together with D is equal to L ; (constr.)

*therefore EG is greater than L ;*

but FG is not greater than L ; (constr.)

and EG, FG, were proved to be equimultiples of AB, BC ;

and L is a multiple of D ; (constr.)

**therefore AB has to D a greater ratio than BC has to D.**

(v. def. 7.)

Also D shall have to BC a greater ratio than it has to AB ;

for the same construction being made, it may be shown, in like manner,

*that L is greater than FG, but that it is not greater than EG ;*

and L is a multiple of D ; (constr.)

and FG, EG, were proved to be equimultiples of CB, AB ;

**therefore D has to CB a greater ratio than it has to AB.**

(v. def. 7.)

Wherefore, of unequal magnitudes, &c

Q. E. D.

---

PROP. IX.—THEOREM.

*Magnitudes which have the same ratio to the same magnitude are equal to one another ; and those to which the same magnitude has the same ratio are equal to one another.*

(Reference—v. def. 5.)

Let A and B have each of them the same ratio to C.

**Then A shall be equal to B.**

|         |         |         |         |
|---------|---------|---------|---------|
| A _____ |         | D _____ |         |
|         | C _____ |         | F _____ |
| B _____ |         | E _____ |         |

## CONSTRUCTION

For, if they are not equal, one of them must be greater than the other ;  
let A be the greater ;

then, by what was shown in the preceding proposition, there are some equimultiples of A and B, and some multiple of C, such, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than that of C ;

let these multiples be taken ; and let D, E be the equimultiples of A, B, and F the multiple of C, so that D may be greater than F, but E not greater than F.

## DEMONSTRATION

Then, because A is to C as B is to C, (hyp.)

and of A, B, are taken equimultiples D, E, and of C is taken a multiple F ;

and that D is greater than F ;

*therefore E must also be greater than F ; (v. def. 5.)*

but E is not greater than F ; (constr.) which is impossible ;

**therefore A and B are not unequal ; that is, they are equal.**

*Next, let C have the same ratio to each of the magnitudes A and B.*

**Then A shall be equal to B.**

## CONSTRUCTION

For, if they are not equal, one of them must be greater than the other ;  
let A be the greater ;

then, as was shown in Prop. VIII., there is some multiple F of C, and some equimultiples E and D of B and A such, that F is greater than E, but not greater than D ;

let these multiples be taken.

## DEMONSTRATION

And because C is to B, as C is to A ; (hyp.)

and that F the multiple of the first, is greater than E the multiple of the second ;

*therefore F the multiple of the third, must be greater than D the multiple of the fourth ; (v. def. 5.)*

but F is not greater than D ; (hyp.) which is impossible ;

**therefore A is equal to B.**

Wherefore, magnitudes which, &c.

Q. E. D.

## PROP. X.—THEOREM.

*That magnitude which has a greater ratio than another has to the same magnitude, is the greater of the two;*

*and that magnitude to which the same has a greater ratio than it has to another magnitude, is the less of the two.*

(References—v. def. 7 ; ax. 4.)

*First, let A have to C a greater ratio than B has to C.*

**Then A shall be greater than B.**

|         |         |
|---------|---------|
| A _____ | D _____ |
| B _____ | E _____ |
| C _____ | F _____ |

## CONSTRUCTION

For, because A has a greater ratio to C, than B has to C, there are some equimultiples of A and B, and some multiple of C such, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than it; (v. def. 7.)

let them be taken ; and let D, E be the equimultiples of A and B, and F the multiple of C ; such, that D is greater than F, but E is not greater than F. Then D is greater than E.

## DEMONSTRATION

Because D and E are equimultiples of A and B, and that D is greater than E ; (constr.)

**therefore A is greater than B. (v. ax. 4.)**

*Next, let C have a greater ratio to B than it has to A.*

**Then B shall be less than A.**

## CONSTRUCTION]

For there is some multiple F of C, and some equimultiples E and D of B and A such, that F is greater than E, but not greater than D ; (v. def. 7.)

let them be taken. Then E is less than D.

## DEMONSTRATION

And because E and D are equimultiples of B and A, and that E is less than D ; (constr.)

**therefore B is less than A. (v. ax. 4.)**

Therefore, that magnitude, &c.

Q. E. D.

## PROP. XL.—THEOREM.

*Ratios that are the same to the same ratio, are the same to one another.*

(Reference—v. def. 5.)

Let A be to B as C is to D, and let C be to D as E is to F.

**Then A shall be to B, as E to F.**

|         |         |         |
|---------|---------|---------|
| G _____ | H _____ | K _____ |
| A _____ | C _____ | E _____ |
| B _____ | D _____ | F _____ |
| L _____ | M _____ | N _____ |

## CONSTRUCTION

Take of A, C, E, any equimultiples whatever G, H, K ; and of B, D, F any equimultiples whatever L, M, N.

## DEMONSTRATION

Then because A is to B as C to D

and G, H are taken equimultiples of A, C, and L, M, of B, D ;

*if G be greater than L, H is greater than M; and if equal, equal; and if less, less; (v. def. 5.)*

Again, because C is to D, as E is to F,

and H, K are taken equimultiples of C, E, and M, N, of D, F ;

*if H be greater than M, K is greater than N; and if equal, equal; and if less, less;*

but if G be greater than L, it has been shown that H is greater than M ; and if equal, equal ; and if less, less ;

*therefore, if G be greater than L, K is greater than N; and if equal, equal ; and if less, less;*

and G, K are any equimultiples whatever of A, E, and L, N any whatever of B, F ;

**therefore, as A is to B, so is E to F. (v. def. 5.)**

Wherefore, ratios that, &c.

Q. E. D.

## PROP. XII.—THEOREM.

*If any number of magnitudes be proportionals ;  
then as one of the antecedents is to its consequent, so shall all the ante-  
cedents taken together be to all the consequents.*

(References—Prop. v. 1, def. 5.)

Let any number of magnitudes A, B, C, D, E, F, be proportionals ;  
that is, as A is to B, so C to D, and E to F.

**Then as A is to B, so shall A, C, E together, be to B, D, F together.**

|         |         |         |
|---------|---------|---------|
| G _____ | H _____ | K _____ |
| A _____ | C _____ | E _____ |
| B _____ | D _____ | F _____ |
| L _____ | M _____ | N _____ |

## CONSTRUCTION

Take of A, C, E, any equimultiples whatever G, H, K ; and of B, D, F any equimultiples whatever L, M, N.

## DEMONSTRATION

Then, because A is to B, as C is to D, and as E to F ;  
and that G, H, K are equimultiples of A, C, E, and L, M, N equimultiples of B, D, F ;

therefore, if G be greater than L, H is greater than M, and K greater than N ; and if equal, equal ; and if less, less ; (v. def. 5.)

*wherefore, if G be greater than L, then G, H, K together, are greater than L, M, N together ; and if equal, equal ; and if less, less ;*

but if there be any number of magnitudes equimultiples of as many, each of each, whatever multiple one of them is of its part, the same multiple is the whole of the whole ; (v. 1.)

*therefore G, and G, H, K together, are equimultiples of A, and A, C, E together ;*

for the same reason L, and L, M, N are equimultiples of B, and B, D, F ;

**therefore as A is to B, so are A, C, E together to B, D, F together.** (v. def. 5.)

Wherefore, if any number, &c.

Q. E. D.

## PROP. XIII.—THEOREM.

*If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth;*

*then the first shall also have to the second a greater ratio than the fifth has to the sixth.*

(References—v. def. 5, 7.)

Let A the first have the same ratio to B the second, which C the third has to D the fourth, but C the third a greater ratio to D the fourth, than E the fifth has to F the sixth.

**Then also the first A shall have to the second B, a greater ratio than the fifth E has to the sixth F.**

|         |         |         |
|---------|---------|---------|
| M _____ | G _____ | H _____ |
| A _____ | C _____ | E _____ |
| B _____ | D _____ | F _____ |
| N _____ | K _____ | L _____ |

## CONSTRUCTION

Because C has a greater ratio to D, than E to F, there are some equimultiples of C and E, and some of D and F such, that the multiple of C is greater than the multiple of D, but the multiple of E is not greater than the multiple of F; (v. def. 7.)

let these be taken, and let G, H be equimultiples of C, E, and K, L equimultiples of D, F, so that G may be greater than K, but H not greater than L;

and whatever multiple G is of C, take M the same multiple of A;

and whatever multiple K is of D, take N the same multiple of B.

## DEMONSTRATION

Then, because A is to B, as C to D; (hyp.)

and of A and C, M and G are equimultiples, and of B and D, N and K are equimultiples;

*therefore, if M be greater than N, G is greater than K; and if equal, equal; and if less, less; (v. def. 5.)*

but G is greater than K; (constr.)

*therefore M is greater than N;*

but H is not greater than L; (constr.)

and M, H are equimultiples of A, E, and N, L equimultiples of B, F;

**therefore A has a greater ratio to B, than E has to F.**

(v. def. 7.)

Wherefore, if the first, &c.

Q. E. D.

Cor. And if the first have a greater ratio to the second, than the third has to the fourth, but the third the same ratio to the fourth, which the fifth has to the sixth ; it may be demonstrated, in like manner, that the first has a greater ratio to the second, than the fifth has to the sixth.

---

PROP. XIV.—THEOREM.

*If the first has the same ratio to the second which the third has to the fourth ;*

*then, if the first be greater than the third, the second shall be greater than the fourth ; and if equal, equal ; and if less, less.*

(References—Prop. v. 8, 9, 10, 13.)

Let the first A have the same ratio to the second B which the third C has to the fourth D.

**If A be greater than C, B shall be greater than D ;**

**and if A be equal to C, B shall be equal to D ;**

**and if A be less than C, B shall be less than D.**

| 1.      | 2.      | 3.      |
|---------|---------|---------|
| A _____ | A _____ | A _____ |
| B _____ | B _____ | B _____ |
| C _____ | C _____ | C _____ |
| D _____ | D _____ | D _____ |

*First, let A be greater than C. Then B shall be greater than D.*

DEMONSTRATION

Because A is greater than C, and B is any other magnitude,  
*therefore A has to B a greater ratio than C has to B ; (v. 8.)*

but, as A is to B, so is C to D ; (hyp.)

*therefore also C has to D a greater ratio than C has to B ; (v. 13.)*

but of two magnitudes, that to which the same has the greater ratio is the less ; (v. 10.)

*therefore D is less than B ;*

**that is, B is greater than D.**

*Secondly, let A be equal to C. Then B shall be equal to D.*

For A is to B, as C, that is, A to D ;

**therefore B is equal to D. (v. 9.)**

*Thirdly, let A be less than C. Then B shall be less than D.*

For C is greater than A ;

and because C is to D, as A is to B, (hyp.)

*therefore D is greater than B, by the first case ;*

**that is, B is less than D.**

*Therefore, if the first, &c.*

Q. E. D.



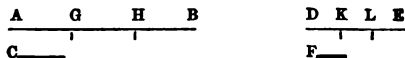
## PROP. XV.—THEOREM.

*Magnitudes have the same ratio to one another which their equimultiples have.*

(References—Prop. v. 7, 12.)

Let AB be the same multiple of C, that DE is of F.

Then C shall be to F, as AB to DE.



## CONSTRUCTION

Because AB is the same multiple of C, that DE is of F, there are as many magnitudes in AB equal to C, as there are in DE equal to F; let AB be divided into magnitudes, each equal to C, viz. AG, GH, HB; and DE into magnitudes, each equal to F, viz. DK, KL, LE.

## DEMONSTRATION

Then the number of the first AG, GH, HB, is equal to the number of the last DK, KL, LE;

and because AG, GH, HB are all equal;

and that DK, KL, LE are also equal to one another;

*therefore, AG is to DK as GH to KL, and as HB to LE; (v. 7.)*

but as one of the antecedents is to its consequent, so are all the antecedents together to all the consequents together; (v. 12.)

*wherefore, as AG is to DK, so is AB to DE;*

but AG is equal to C, and DK to F,

**therefore, as C is to F, so is AB to DE.**

Therefore, magnitudes, &c.

Q. E. D.

## PROP. XVI.—THEOREM.

*If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.*

(References—Prop. v. 11, 14, 15, def. 5.)

Let A, B, C, D be four magnitudes of the same kind, which are proportionals, viz. as A to B, so C to D.

**They shall also be proportionals when taken alternately,  
that is A shall be to C, as E to D.**

|         |         |
|---------|---------|
| E _____ | G _____ |
| A _____ | C _____ |
| B _____ | D _____ |
| F _____ | H _____ |

## CONSTRUCTION

Take of A and B any equimultiples whatever E and F ; and of C and D take any equimultiples whatever G and H.

## DEMONSTRATION

Because E is the same multiple of A, that F is of B,  
and that magnitudes have the same ratio to one another which their  
equimultiples have ; (v. 15.)

*therefore A is to B, as E is to F ;*

but as A is to B, so is C to D ; (hyp.)

*wherefore, as C is to D, so is E to F ;* (v. 11.)

Again, because G, H are equimultiples of C, D,

*therefore, as C is to D, so is G to H ;* (v. 15.)

but it was proved that as C is to D, so is E to F ;

*therefore, as E is to F, so is G to H ;* (v. 11.)

but when four magnitudes are proportionals, if the first be greater than  
the third, the second is greater than the fourth ; and if equal, equal ;  
and if less, less ; (v. 14.)

*therefore, if E be greater than G, F likewise is greater than H ; and if  
equal, equal ; and if less, less ;*

and E, F are any equimultiples whatever of A, B, and G, H any what-  
ever of C, D ; (constr.)

**therefore A is to C as B to D.** (v. def. 5.)

If then four magnitudes, &c.

Q. E. D.

## PROP. XVII.—THEOREM.

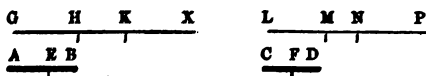
*If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately; that is, if two magnitudes together have to one of them, the same ratio which two others have to one of these;*

*then the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.*

(References—Prop. v. 1, 2, def. 5.)

Let AB, BE, CD, DF be the magnitudes, taken jointly, which are proportionals; that is, as AB to BE, so let CD be to DF.

Then they shall also be proportionals taken separately, viz. as AE to EB, so shall CF be to FD.



## CONSTRUCTION

Take of AE, EB, CF, FD any equimultiples whatever GH, HK, LM, MN; and again, of EB, FD take any equimultiples whatever KX, NP.

## DEMONSTRATION

Then because GH is the same multiple of AE, that HK is of EB,

*therefore GH is the same multiple of AE, that GK is of AB; (v. 1.)*

but GH is the same multiple of AE, that LM is of CF;

*therefore GK is the same multiple of AB, that LM is of CF.*

Again, because LM is the same multiple of CF, that MN is of FD;

*therefore LM is the same multiple of CF, that LN is of CD; (v. 1.)*

but LM was shown to be the same multiple of CF, that GK is of AB;

*therefore GK is the same multiple of AB, that LN is of CD;*

*that is GK, LN are equimultiples of AB, CD.*

Next, because HK is the same multiple of EB, that MN is of FD;

and that KX is also the same multiple of EB, that NP is of FD;

*therefore HX is the same multiple of EB, that MP is of FD; (v. 2.)*

And because AB is to BE, as CD is to DF; (hyp.)

and that of AB and CD, GK and LN are equimultiples;

and of EB and FD, HX and MP are equimultiples;

*therefore if GK be greater than HX, then LN is greater than MP; and*

*if equal, equal; and if less, less; (v. def. 5.)*

but if GH be greater than KX, then, by adding the common part HK to both, GK is greater than HX ; (1. ax. 4.)

*wherefore also LN is greater than MP;*

and by taking away MN from both, LM is greater than NP; (1. ax. 5.)

*therefore, if GH be greater than KX, LM is greater than NP.*

In like manner it may be demonstrated, that if GH be equal to KX, LM is equal to NP; and if less, less;

but GH, LM are any equimultiples whatever of AE, CF, and KX, NP are any whatever of EB, FD ; (constr.)

**therefore, as AE is to EB, so is CF to FD.** (v. def. 5.)

If then magnitudes, &c.

Q. E. D.

PROP. XVIII.—THEOREM.

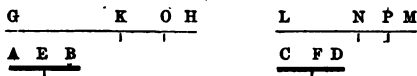
*If magnitudes, taken separately, be proportionals, they shall also be proportionals when taken jointly; that is, if the first be to the second, as the third to the fourth;*

*then the first and second together shall be to the second, as the third and fourth together to the fourth.*

(References—Prop. v. 5, 6, 4 cor., A, def. 5, ax. 3.

Let AE, EB, CF, FD be proportionals; that is, as AE to EB, so let CF be to FD.

**Then they shall also be proportionals when taken jointly; that is, as AB to BE, so shall CD be to DF.**



CONSTRUCTION

Take of AB, BE, CD, DF any equimultiples whatever GH, HK, LM, MN;

and again, of BE, DF, take any equimultiples whatever KO, NP.

DEMONSTRATION

And because KO, NP are equimultiples of BE, DF, and that KH, NM are likewise equimultiples of BE, DF;

if then KO, the multiple of BE, be greater than KH, which is a multiple of the same BE,

*therefore NP, the multiple of DF, is also greater than NM, the multiple of the same DF;*

and if KO be equal to KH, NP is equal to NM; and if less, less.

First, let KO be not greater than KH ;

therefore NP is not greater than NM ;

and because GH, HK, are equimultiples of AB, BE, and that AB is greater than BE,

therefore GH is greater than HK ; (v. ax. 3.)

but KO is not greater than HK ;

therefore GH is greater than KO.

In like manner it may be shown, that LM is greater than NP ;

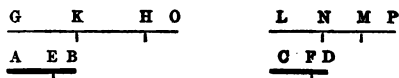
if then, KO be not greater than KH,

therefore GH, the multiple of AB, is always greater than KO, the multiple of BE ;

and likewise LM, the multiple of CD, is greater than NP, the multiple of DF.

Next, let KO be greater than KH ;

therefore, as has been shown, NP is greater than NM.



And because the whole GH is the same multiple of the whole AB, that HK is of BE,

therefore the remainder GK is the same multiple of the remainder AE that GH is of AB (v. 5), which is the same that LM is of CD.

In like manner, because LM is the same multiple of CD, that MN is of DF,

therefore the remainder LN is the same multiple of the remainder CF, that the whole LM is of the whole CD ; (v. 5.)

but it was shown that LM is the same multiple of CD, that GK is of AE ;

therefore GK is the same multiple of AE, that LN is of CF ;

that is, GK, LN are equimultiples of AE, CF.

And because KO, NP are equimultiples of BE, DF ;

if from KO, NP there be taken KH, NM, which are likewise equimultiples of BE, DF,

therefore the remainders HO, MP are either equal to BE, DF, or equimultiples of them ; (v. 6.)

First, let HO, MP be equal to BE, DF ;

then because AE is to EB, as CF to FD ; (hyp.)

and that GK, LN are equimultiples of AE, CF ;

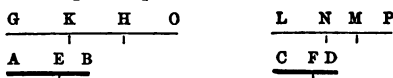
*therefore GK is to EB, as LN to FD; (v. 4. cor.)*

but HO is equal to EB, and MP to FD;

wherefore GK is to HO, as LN to MP;

*therefore if GK be greater than HO, LN is greater than MP; and if equal, equal; and if less, less; (v. A.)*

But let HO, MP be equimultiples of EB, FD.



Then because AE is to EB, as CF to FD, (hyp.)

and that of AE, CF are taken equimultiples GK, LN;

and of EB, FD, the equimultiples HO, MP;

*if GK be greater than HO, LN is greater than MP; and if equal, equal; and if less, less; (v. def. 5.)*

which was likewise shown in the preceding case.

But if GH be greater than KO, taking KH from both, GK is greater than HO; (I. ax. 5.)

wherefore also LN is greater than MP;

and, consequently, adding NM to both, LM is greater than NP; (I. ax. 4.)

*therefore, if GH be greater than KO, LM is greater than NP.*

In like manner it may be shown, that if GH be equal to KO, LM is equal to NP; and if less, less.

And in the case in which KO is not greater than KH, it has been shown that GH is always greater than KO, and likewise LM greater than NP;

but GH, LM are any equimultiples whatever of AB, CD, and KO, NP are any whatever of BE, DF; (constr.)

**therefore, as AB is to BE, so is CD to DF, (v. def. 5.)**

If then magnitudes, &c.

Q. E. D.

### PROP. XIX.—THEOREM.

*If a whole magnitude be to a whole, as a magnitude taken from the first is to a magnitude taken from the other;*

*then the remainder shall be to the remainder as the whole to the whole.*

(References—Prop. v. 11, 16, 17.)

Let the whole AB be to the whole CD, as AE a magnitude taken from AB is to CF a magnitude taken from CD.

**Then the remainder EB shall be to the remainder FD, as the whole AB to the whole CD.**

$$\begin{array}{ccccc} A & E & & B & \\ \hline & | & & & \\ C & F & & D & \\ \hline \end{array}$$

#### DEMONSTRATION

Because AB is to CD, as AE to CF ;

*therefore alternately, BA is to AE, as DC to CF; (v. 16.)*

and because if magnitudes taken jointly be proportionals, they are also proportionals, when taken separately; (v. 17.)

therefore, as BE is to EA, so is DF to FC ;

*and alternately, as BE is to DF, so is EA to FC;*

but, as AE to CF, so by the hypothesis, is AB to CD ;

**therefore also EB the remainder is to the remainder DF, as the whole AB to the whole CD. (v. 11.)**

Wherefore, if the whole, &c.

Q. E. D.

Cor. If the whole be to the whole, as a magnitude taken from the first is to a magnitude taken from the other; the remainder shall likewise be to the remainder, as the magnitude taken from the first to that taken from the other. The demonstration is contained in the preceding.

#### PROP. E.—THEOREM.

*If four magnitudes be proportionals, they are also proportionals by conversion ;*

*that is, the first is to its excess above the second, as the third to its excess above the fourth.*

(References—Prop. v. B, 17, 18.)

Let AB be to BE, as CD to DF.

**Then EA shall be to AE, as DC to CF.**

$$\begin{array}{ccccc} A & & E & B & \\ \hline & & | & & \\ C & & F & D & \\ \hline \end{array}$$

## DEMONSTRATION

Because AB is to BE, as CD to DF,

*therefore by division, AE is to EB, as CF to FD; (v. 17.)*

and by inversion, BE is to EA, as DF to FC; (v. B.)

**wherefore, by composition, EA is to AE, as DC is to CF. (v. 18.)**

If therefore four, &c.

Q. E. D.

## PROP. XX.—THEOREM.

*If there be three magnitudes, and other three, which, taken two and two, have the same ratio;*

*then if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.*

(References—Prop. v. B, 7, 8, 9, 10, 11, 13 cor.)

Let A, B, C, be three magnitudes, and D, E, F, other three, which taken two and two have the same ratio, viz. as A is to B, so is D to E, and as B to C, so is E to F,

**Then if A be greater than C, D shall be greater than F.**

**and if A be equal to C, D shall be equal to F.**

**and if A be less than C, D shall be less than F.**

|         |         |         |
|---------|---------|---------|
| A _____ | B _____ | C _____ |
| D _____ | E _____ | F _____ |

*First, let A be greater than C. Then D shall be greater than F.*

## DEMONSTRATION

Because A is greater than C, and B is any other magnitude, and that the greater has to the same magnitude a greater ratio than the less has to it; (v. 8.)

*therefore A has to B a greater ratio than C has to B;*

but as D is to E, so is A to B; (hyp.)

*therefore D has to E a greater ratio than C to B; (v. 13.)*

and because B is to C, as E to F,

*by inversion, C is to B, as F is to E; (v. B.)*

and D was shown to have to E a greater ratio than C to B;



*therefore D has to E a greater ratio than F to E; (v. 13 cor.)*  
 but the magnitude which has a greater ratio than another to the same  
 magnitude, is the greater of the two; (v. 10.)

**therefore D is greater than F.**

*Secondly*, let A be equal to C. Then D shall be equal to F.

|        |        |        |
|--------|--------|--------|
| A_____ | B_____ | C_____ |
| D_____ | E_____ | F_____ |

#### DEMONSTRATION

Because A and C are equal to one another,  
*therefore A is to B, as C is to B; (v. 7.)*  
 but A is to B, as D to E; (hyp.)  
 and C is to B, as F to E; (hyp.)  
*wherefore D is to E, as F to E; (v. 11, and v. B.)*  
**therefore D is equal F. (v. 9.)**

*Thirdly*, let A be less than C. Then D shall be less than F.

|        |        |        |
|--------|--------|--------|
| A_____ | B_____ | C_____ |
| D_____ | E_____ | F_____ |

#### DEMONSTRATION

For C is greater than A;  
 and as was shown in the first case, C is to B, as F to E,  
 and in like manner, B is to A, as E to D;  
*therefore F is greater than D, by the first case;*  
**that is, D is less than F.**

Therefore, if there be three, &c.

Q. E. D.

#### PROP. XXI.—THEOREM.

*If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order;  
 then if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.*

(References—Prop. v. 7, 8, 9, 10, 11, 13 cor.)

Let A, B, C be three magnitudes, and D, E, F other three, which have the same ratio, taken two and two, but in a cross order; viz. as A is to B, so is E to F, and as B is to C, so is D to E.

**Then if A be greater than C, D shall be greater than F ;**

**and if A be equal to C, D shall be equal to F ;**

**and if A be less than C, D shall be less than F.**

|         |         |         |
|---------|---------|---------|
| A _____ | B _____ | C _____ |
| D _____ | E _____ | F _____ |

*First*, let A be greater than C. Then D shall be greater than F.

#### DEMONSTRATION

Because A is greater than C, and B is any other magnitude,  
*therefore A has to B a greater ratio than C has to B ; (v. 8.)*

but as E to F, so is A to B ; (hyp.)

*therefore E has to F a greater ratio than C to B ; (v. 13.)*

and because B is to C, as D to E; (hyp.)

*by inversion, C is to B, as E to D ;*

and E was shown to have to F a greater ratio than C has to B ;

*therefore E has to F a greater ratio than E has to D ; (v. 13, cor.)*

but the magnitude to which the same has a greater ratio than it has to another, is the less of the two ; (v. 10.)

*therefore F is less than D ;*

**that is, D is greater than F.**

*Secondly*, let A be equal to C. Then D shall be equal to F.

|         |         |         |
|---------|---------|---------|
| A _____ | B _____ | C _____ |
| D _____ | E _____ | F _____ |

#### DEMONSTRATION

Because A and C are equal,

*therefore A is to B, as C is to B ; (v. 7.)*

but A is to B, as E to F; (hyp.)

and C is to B, as E to D ;

*therefore E is to F, as E to D ; (v. 11.)*

**wherefore D is equal to F. (v. 9.)**

*Thirdly*, let A be less than C. Then D shall be less than F.

|         |         |         |
|---------|---------|---------|
| A _____ | B _____ | C _____ |
| D _____ | E _____ | F _____ |

## DEMONSTRATION

For  $C'$  is greater than  $A$  ;  
 and, as was shown,  $C$  is to  $B$ , as  $E$  to  $D$ ,  
 and in like manner,  $B$  is to  $A$ , as  $F$  to  $E$ ;  
*therefore  $F$  is greater than  $D$ , by the first case*  
**that is,  $D$  is less than  $F$ .**  
 Therefore, if there be three, &c.

Q. E. D.

## PROP. XXII.—THEOREM.

*If there be any number of magnitudes, and as many others, which taken two and two in order, have the same ratio ;  
 then the first shall have to the last of the first magnitudes, the same ratio which the first of the others has to the last.*

N.B. This is usually cited by the words 'Ex æquali,' or 'Ex æquo.'

(References—Prop. v. 4, 20, def. 5.)

*First*, let there be three magnitudes  $A, B, C$ , and as many others  $D, E, F$ , which taken two and two in order, have the same ratio; that is, such that  $A$  is to  $B$  as  $D$  to  $E$ ; and as  $B$  is to  $C$ , so is  $E$  to  $F$ :

**Then  $A$  shall be to  $C$ , as  $D$  to  $F$ .**

|           |           |           |
|-----------|-----------|-----------|
| $G$ _____ | $K$ _____ | $M$ _____ |
| $A$ _____ | $B$ _____ | $C$ _____ |
| $D$ _____ | $E$ _____ | $F$ _____ |
| $H$ _____ | $L$ _____ | $N$ _____ |

## CONSTRUCTION

Take of  $A$  and  $D$  any equimultiples whatever  $G$  and  $H$  ;  
 and of  $B$  and  $E$  any equimultiples whatever  $K$  and  $L$  ;  
 and of  $C$  and  $F$  any whatever  $M$  and  $N$ .

## DEMONSTRATION

Then because  $A$  is to  $B$ , as  $D$  to  $E$  ; (hyp.)  
 and that  $G, H$  are equimultiples of  $A, D$ , and  $K, L$ , equimultiples of  $B, E$ ;  
*therefore as  $G$  is to  $K$ , so is  $H$  to  $L$  ; (v. 4.)*  
 for the same reason,  $K$  is to  $M$  as  $L$  to  $N$ .  
 And because there are three magnitudes  $G, K, M$ , and other three  $H, L, N$ , which two and two, have the same ratio ;

*therefore if G be greater than M, H is greater than N ; and if equal, equal; and if less, less ; (v. 20.)*

but G, H are any equimultiples whatever of A, D, and M, N are any equimultiples whatever of C, F ; (constr.)

**therefore, as A is to C, so is D to F.** (v. def. 5.)

*Next*, let there be four magnitudes, A, B, C, D, and other four E, F, G, H, which two and two have the same ratio, viz., as A is to B, so is E to F ; and as B to C, so is F to G ; and as C to D, so G to H.

**Then A shall be to D, as E to H.**

|               |
|---------------|
| A . B . C . D |
| E . F . G . H |

#### DEMONSTRATION

Because A, B, C are three magnitudes, and E, F, G, other three, which taken two and two, have the same ratio;

*therefore by the foregoing case, A is to C, as E to G ;*

but C is to D, as G to H ;

**wherefore again by the first case, A is to D, as E to H ;**

and so on, whatever be the number of magnitudes.

Therefore, if there be any number, &c.

Q. E. D.

#### PROP. XXIII.—THEOREM.

*If there be any number of magnitudes, and as many others, which taken two and two in a cross order, have the same ratio ;*

*then the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.*

N.B. This is usually cited by the words 'Ex aequali in proportione perturbatâ,' or 'Ex æquo perturbato.'

(References—Prop. v. 4, 11, 15, 21, def. 5.)

*First*, let there be three magnitudes A, B, C, and other three, D, E, F, which taken two and two in a cross order have the same ratio, that is, such that A is to B, as E to F, and as B is to C, so is D to E.

**Then A shall be to C, as D to F.**

|        |        |        |
|--------|--------|--------|
| G_____ | H_____ | L_____ |
| A_____ | B_____ | C_____ |
| D_____ | E_____ | F_____ |
| K_____ | M_____ | N_____ |

## CONSTRUCTION

Take of A, B, D any equimultiples whatever G, H, K;  
and of C, E, F any equimultiples whatever L, M, N.

## DEMONSTRATION

Then because G, H are equimultiples of A, B,  
and that magnitudes have the same ratio which their equimultiples  
have ; (v. 15.)

*therefore as A is to B, so is G to H ;*

for the same reason, as E is to F, so is M to N ;

but as A is to B, so is E to F ; (hyp.)

*therefore as G is to H, so is M to N ;* (v. 11.)

And because as B is to C, so is D to E; (hyp.)

and that H, K are equimultiples of B, D, and L, M of C, E;

*therefore as H is to L, so is K to M ;* (v. 4.)

and it has been shown that G is to H, as M to N ;

then, since there are three magnitudes G, H, L, and other three K, M,  
N, which have the same ratio taken two and two in a cross order;

*therefore if G be greater than L, K is greater than N; and if equal,  
equal ; and if less, less ;* (v. 21.)

but G, K are equimultiples whatever of A, D ; and L, N any whatever  
of C, F ; (constr.)

**therefore as A is to C, so is D to F.** (v. def. 5.)

*Next*, let there be four magnitudes A, B, C, D, and other four E, F, G,  
H, which taken two and two in a cross order have the same ratio,  
viz. A to B, as G to H ; B to C, as F to G ; and C to D, as E to F.

**Then A shall be to D, as E to H.**

|               |
|---------------|
| A . B . C . D |
| E . F . G . H |

## DEMONSTRATION

Because A, B, C are three magnitudes, and F, G, H other three, which  
taken two and two in a cross order, have the same ratio ;

*therefore, by the first case, A is to C, as F to H ;*

but C is to D, as E is to F ;

**wherefore again, by the first case, A is to D, as E to H ;**

and so on, whatever be the number of magnitudes.

Therefore, if there be any number, &c.

Q. E. D.

PROP. XXIV.—THEOREM.

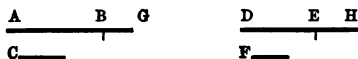
*If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second the same ratio which the sixth has to the fourth;*

*then the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.*

(References—Prop. v. B, 18, 22.)

Let AB the first have to C the second the same ratio which DE the third has to F the fourth; and let BG the fifth have to C the second the same ratio which EH the sixth has to F the fourth.

**Then AG, the first and fifth together, shall have to C the second, the same ratio which DE, the third and sixth together, has to F the fourth.**



DEMONSTRATION

Because BG is to C, as EH to F;

*therefore, by inversion, C is to BG, as F to EH; (v. B.)*

and because, as AB is to C, so is DE to F; (hyp.)

and as C to BG, so is F to EH;

*therefore, ex æquali, AB is to BG, as DE to EH; (v. 22.)*

and because these magnitudes are proportionals when taken separately, they are likewise proportionals when taken jointly; (v. 18.)

*therefore as AG is to GB, so is DH to HE;*

but as GB to C, so is HE to F; (hyp.)

**therefore ex sequenti, as AG is to C, so is DH to F.**  
(v. 22.)

Wherefore, if the first, &c.

Q. E. D.

Cor. 1. If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second, as the excess of the third and sixth to the fourth. The demonstration of this is the same with that of the proposition, if division be used instead of composition.

Cor. 2. The proposition holds true of two ranks of magnitudes whatever be their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude; as is manifest.

## PROP. XXV.—THEOREM.

*If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.*

(References—Prop. v. A, 7, 11, 14, 19.)

Let the four magnitudes AB, CD, E, F be proportionals, viz. AB to CD, as E to F; and let AB be the greatest of them, and consequently F the least. (v. 14 & A.)

**Then AB together with F shall be greater than CD together with E.**



## CONSTRUCTION

Take AG equal to E, and CH equal to F.

## DEMONSTRATION

Then because as AB is to CD, so is E to F, (hyp.)

and that AG is equal to E, and CH equal to F,

*therefore AB is to CD, as AG to CH; (v. 11 & 7.)*

and because AB the whole, is to the whole CD, as AG is to CH,

*therefore the remainder GB is to the remainder HD, as the whole AB is to the whole CD; (v. 19.)*

but AB is greater than CD; (hyp.)

*therefore GB is greater than HD; (v. A.)*

and because AG is equal to E, and CH to F; (constr.)

*therefore AG and F together are equal to CH and E together;*  
(I. ax. 2.)

if then to the unequal magnitudes GB, HD, of which GB is the greater, there be added equal magnitudes, viz. AG and F to GB, and CH and E to HD;

**therefore AB and F together are greater than CD and E.**  
(I. ax. 4.)

Therefore, if four magnitudes, &c.

Q. E. D.

## PROP. F.—THEOREM.

*Ratios which are compounded of the same ratios are the same to one another.*

(References—Prop. v. 22, 23.)

Let A be to B, as D to E; and B to C, as E to F.

**Then the ratio which is compounded of the ratios of A to B, and B to C, which by the definition of compound ratio, is the ratio of A to C, shall be the same with the ratio of D to F, which, by the same definition, is compounded of the ratios of D to E, and E to F.**

|           |
|-----------|
| A . B . C |
| D . E . F |

## DEMONSTRATION

Because there are three magnitudes A, B, C, and three others, D, E, F, which, taken two and two, in order, have the same ratio;

**therefore, ex æquali, A is to C, as D to F. (v. 22.)**

Next, let A be to B, as E to F, and B to C, as D to E;

|           |
|-----------|
| A . B . C |
| D . E . F |

*therefore, ex æquali in proportionibus perturbatâ, A is to C, as D to F;*  
(v. 23.)

**that is, the ratio of A to C, which is compounded of the ratios of A to B, and B to C, is the same with the ratio of D to F, which is compounded of the ratios of D to E, and E to F.**

And in like manner the proposition may be demonstrated, whatever be the number of ratios in either case.

## PROP. G.—THEOREM.

*If several ratios be the same to several ratios, each to each; then the ratio which is compounded of ratios which are the same to the first ratios, each to each; shall be the same to the ratio compounded of ratios which are the same to the other ratios, each to each.*

(Reference—Prop. v. 22.)



Let A be to B, as E to F ; and C to D, as G to H ; and let A be to B, as K to L ; and C to D, as L to M ;

then the ratio of K to M, by the definition of compound ratio, is compounded of the ratios of K to L, and L to M, which are the same with the ratios of A to B and C to D.

Again, as E to F, so let N be to O ; and as G to H, so let O be to P ;

then the ratio of N to P is compounded of the ratios of N to O, and O to P, which are the same with the ratios of E to F, and G to H.

**And it is to be shown that the ratio of K to M, is the same with the ratio of N to P ; or that K is to M, as N to P.**

|                           |
|---------------------------|
| A . B . C . D . K . L . M |
| E . F . G . H . N . O . P |

#### DEMONSTRATION

Because K is to L, as (A to B, that is, as E to F, that is, as) N to O ; and as L to M, so is (C to D, and so is G to H, and so is) O to P ;

**therefore, ex sequenti, K is to M, as N to P. (v. 22.)**

Therefore, if several ratios, &c.

Q. E. D.

#### PROP. H.—THEOREM.

*If a ratio which is compounded of several ratios be the same to a ratio which is compounded of several other ratios ; and if one of the first ratios, or the ratio which is compounded of several of them, be the same to one of the last ratios, or to the ratio which is compounded of several of them ;*

*then the remaining ratio of the first, or, if there be more than one, the ratio compounded of the remaining ratios, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio compounded of these remaining ratios.*

(References—Prop. v. B, 22.)

Let the first ratios be those of A to B, B to C, C to D, D to E, and E to F ; and let the other ratios be those of G to H, H to K, K to L, and L to M ;

also let the ratio of A to F, which is compounded of the first ratios, be the same with the ratio of G to M, which is compounded of the other ratios ; and besides, let the ratio of A to D, which is compounded of the ratios of A to B, B to C, C to D, be the same with the ratio of G to K, which is compounded of the ratios of G to H, and H to K.

**Then the ratio compounded of the remaining first ratios, to wit, of the ratios of D to E, and E to F, which compounded ratio is the ratio of D to F, shall be the same with the ratio of H to M, which is compounded of the remaining ratios of H to I, and I to M of the other ratios.**

|                       |
|-----------------------|
| A . B . C . D . E . F |
| G . H . K . L . M     |

## DEMONSTRATION

Because, by the hypothesis, A is to D, as G to K,  
*therefore, by inversion, D is to A, as K to G; (v. B.)*  
 and as A is to F, so is G to M; (hyp.)

**therefore, ex sequenti, D is to F, as K to M. (v. 22.)**

If, therefore, a ratio which is, &c.

Q. E. D.

## PROP. K.—THEOREM.

*If there be any number of ratios, and any number of other ratios such, that the ratio which is compounded of ratios which are the same to the first ratios, each to each, is the same to the ratio which is compounded of ratios which are the same, each to each, to the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same to several of the first ratios, each to each, be the same to one of the last ratios or to the ratio which is compounded of ratios which are the same, each to each, to several of the last ratios;*

*then the remaining ratio of the first, or, if there be more than one, the ratio which is compounded of ratios which are the same each to each to the remaining ratios of the first, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio which is compounded of ratios which are the same each to each to these remaining ratios.*

(References—Prop. v. B, 11, 22.)

Let the ratios of A to B, C to D, E to F, be the first ratios; and the ratios of G to H, K to L, M to N, O to P, Q to R, be the other ratios; and let A be to B, as S to T; and C to D, as T to V; and E to F, as V to X;

therefore, by the definition of compound ratio, the ratio of S to X is compounded of the ratios of S to T, T to V, and V to X, which are the same to the ratios of A to B, C to D, E to F; each to each.

Also, as  $G$  to  $H$ , so let  $Y$  be to  $Z$ ; and  $K$  to  $L$ , as  $Z$  to  $a$ ;  $M$  to  $N$ , as  $a$  to  $b$ ;  $O$  to  $P$ , as  $b$  to  $c$ ; and  $Q$  to  $R$ , as  $c$  to  $d$ ;

therefore, by the same definition, the ratio of  $Y$  to  $d$  is compounded of the ratios of  $Y$  to  $Z$ ,  $Z$  to  $a$ ,  $a$  to  $b$ ,  $b$  to  $c$ , and  $c$  to  $d$ , which are the same, each to each, to the ratios of  $G$  to  $H$ ,  $K$  to  $L$ ,  $M$  to  $N$ ,  $O$  to  $P$ , and  $Q$  to  $R$ ;

therefore, by the hypothesis,  $S$  is to  $X$  as  $Y$  to  $d$ .

Also, let the ratio of  $A$  to  $B$ , that is the ratio of  $S$  to  $T$ , which is one of the first ratios, be the same to the ratio of  $e$  to  $g$ , which is compounded of the ratios of  $e$  to  $f$ , and  $f$  to  $g$ , which, by the hypothesis, are the same to the ratios of  $G$  to  $H$ , and  $K$  to  $L$ , two of the other ratios;

and let the ratio of  $h$  to  $l$  be that which is compounded of the ratios of  $h$  to  $k$ , and  $k$  to  $l$ , which are the same to the remaining first ratios, viz. of  $C$  to  $D$ , and  $E$  to  $F$ ;

also, let the ratio of  $m$  to  $p$ , be that which is compounded of the ratios of  $m$  to  $n$ ,  $n$  to  $o$ , and  $o$  to  $p$ , which are the same, each to each, to the remaining other ratios, viz. of  $M$  to  $N$ ,  $O$  to  $P$ , and  $Q$  to  $R$ .

**Then the ratio of  $h$  to  $l$  shall be the same to the ratio of  $m$  to  $p$ ; or  $h$  shall be to  $l$ , as  $m$  to  $p$ .**

|                                 |                     |  |
|---------------------------------|---------------------|--|
| $h, k, l.$                      |                     |  |
| $A, B; C, D; E, F.$             | $S, T, V, X.$       |  |
| $G, H; K, L; M, N; O, P; Q, R.$ | $Y, Z, a, b, c, d.$ |  |
| $e, f, g.$                      | $m, n, o, p.$       |  |

#### DEMONSTRATION

Because  $e$  is to  $f$ , as ( $G$  to  $H$ , that is, as)  $Y$  to  $Z$ ; and  $f$  is to  $g$ , as ( $K$  to  $L$ , that is, as)  $Z$  to  $a$ ;

therefore, *ex æquali*,  $e$  is to  $g$ , as  $Y$  to  $a$ ; (v. 22.)

and  $A$  is to  $B$ , that is  $S$  to  $T$ , as  $e$  to  $g$ ; (hyp.)

therefore  $S$  is to  $T$ , as  $Y$  to  $a$ ; (v. 11.)

and by inversion,  $T$  is to  $S$ , as  $a$  to  $Y$ ; (v. B.)

but  $S$  to  $X$ , as  $Y$  to  $D$ ; (hyp.)

therefore, *ex æquali*,  $T$  is to  $X$ , as  $a$  to  $d$ .

Also, because  $h$  is to  $k$ , as ( $C$  to  $D$ , that is, as)  $T$  to  $V$ ; and  $k$  is to  $l$  as ( $E$  to  $F$ , that is, as)  $V$  to  $X$ ;

therefore, *ex æquali*,  $h$  is to  $l$ , as  $T$  to  $X$ .

In like manner, it may be demonstrated, that  $m$  is to  $p$ , as  $a$  to  $d$ ;

and it has been shown, that  $T$  is to  $X$ , as  $a$  to  $d$ ;

therefore  $h$  is to  $l$ , as  $m$  to  $p$ . (v. 11.)

Q. E. D.

## BOOK VI.

## DEFINITIONS.

## I.

SIMILAR rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.



## II.

'Reciprocal figures, viz. triangles and parallelograms, are such as have their sides about two of their angles proportionals in such a manner, that a side of the first figure is to a side of the other, as the remaining side of the other is to the remaining side of the first.'

## III.

A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less.

## IV.

The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.



## PROP. I.—THEOREM.

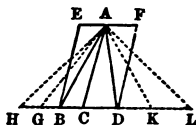
*Triangles and parallelograms of the same altitude, are one to the other as their bases.*

(References—Prop. I. 3, 38, 41 ; v. 11, 15, def. 5.)

Let the triangles ABC, ACD, and the parallelograms EC, CF, have the same altitude, viz. the perpendicular drawn from the point A to BD or BD produced.

**Then as the base BC is to the base CD, so shall the triangle ABC be to the triangle ACD,**

**and the parallelogram EC to the parallelogram CF.**



## CONSTRUCTION

Produce BD both ways to the points H, L,  
and take any number of straight lines BG, GH, each equal to the base BC ; and any number of straight lines DK, KL, each equal to CD ;  
(I. 3.)  
and join AG, AH, AK, AL.

## DEMONSTRATION

Then because CB, BG, GH, are all equal,  
the triangles AHG, AGB, ABC, are all equal ; (I. 38.)  
*therefore, whatever multiple the base HC is of the base BC, the same multiple is the triangle AHC of the triangle ABC ;*  
for the same reason, whatever multiple the base LC is of the base CD,  
the same multiple is the triangle ALC of the triangle ADC.  
And if the base HC be equal to the base CL,  
*the triangle AHC is also equal to the triangle ALC ; (I. 38.)*  
and if the base HC be greater than the base CL,  
*the triangle AHC is likewise greater than the triangle ALC ;*  
and if less, less.  
Since, then, there are four magnitudes, viz the two bases BC, CD, and  
the two triangles ABC, ACD ;

and of the base  $BC$ , and the triangle  $ABC$ , the first and third, any equimultiples whatever have been taken, viz. the base  $HC$  and the triangle  $AHC$ ;

and of the base  $CD$  and the triangle  $ACD$ , the second and fourth, have been taken any equimultiples whatever, viz. the base  $CL$  and the triangle  $ALC$ ;

and since it has been shown, that, if the base  $HC$  be greater than the base  $CL$ , the triangle  $AHC$  is greater than the triangle  $ALC$ ; and if equal, equal; and if less, less;

**therefore, as the base  $BC$  is to the base  $CD$ , so is the triangle  $ABC$  to the triangle  $ACD$ . (v. def. 5.)**

And because the parallelogram  $CE$  is double of the triangle  $ABC$ , and the parallelogram  $CF$  double of the triangle  $ACD$ ; (i. 41.)

and that magnitudes have the same ratio which their equimultiples have; (v. 15.)

*therefore as the triangle  $ABC$  is to the triangle  $ACD$ , so is the parallelogram  $EC$  to the parallelogram  $CF$ ;*

and because it has been shown, that, as the base  $BC$  is to the base  $CD$ , so is the triangle  $ABC$  to the triangle  $ACD$ ;

and as the triangle  $ABC$  is to the triangle  $ACD$ , so is the parallelogram  $EC$  to the parallelogram  $CF$ ;

**therefore, as the base  $BC$  is to the base  $CD$ , so is the parallelogram  $EC$  to the parallelogram  $CF$ . (v. 11.)**

Wherefore, triangles, &c.

Q. E. D.

Cor. From this it is plain, that triangles and parallelograms that have equal altitudes, are to one another as their bases.

For, let the figures be placed so as to have their bases in the same straight line; and having drawn perpendiculars from the vertices of the triangles to the bases, the straight line which joins the vertices is parallel to that in which their bases are (i. 33), because the perpendiculars are both equal and parallel to one another (i. 28). Then, if the same construction be made as in the proposition, the demonstration will be the same.

#### PROP. II.—THEOREM.

*If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, on these produced, proportionally;*

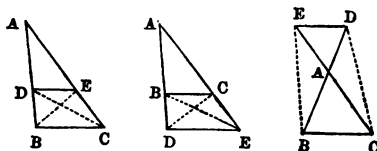
*and conversely, if the sides, or the sides produced, be cut proportionally,*

*the straight line which joins the points of section shall be parallel to the remaining side of the triangle.*

(References—Prop. I. 37, 39 ; v. 7, 9, 11 ; VI. 1.)

Let DE be drawn parallel to BC, one of the sides of the triangle ABC.

**Then BD shall be to DA, as CE to EA.**



#### CONSTRUCTION

Join BE, CD.

#### DEMONSTRATION

Then the triangle BDE is equal to the triangle CDE, because they are on the same base DE, and between the same parallels DE, BC ;  
(I. 37.)

but ADE is another triangle ;

and equal magnitudes have the same ratio to the same magnitude ;  
(v. 7.)

*therefore, as the triangle BDE is to the triangle ADE, so is the triangle CDE to the triangle ADE ;*

but, because the triangles BDE, ADE have the same altitude, viz. the perpendicular drawn from the point E to AB ;

*therefore as the triangle BDE to the triangle ADE, so is BD to DA ;*  
(VI. 1.)

for the same reason, as the triangle CDE to the triangle ADE, so is CE to EA ;

**therefore, as BD to DA, so is CE to EA.** (v. 11.)

*Next, let the sides AB, AC of the triangle ABC, or these sides produced, be cut proportionally in the points D, E, that is, so that BD may be to DA as CE to EA, and join DE.*

**Then DE shall be parallel to BC.**

#### DEMONSTRATION

The same construction being made,  
*because as BD to DA, so is CE to EA ;*

and as BD to DA, so is the triangle BDE to the triangle ADE; (vi. 1.)  
and as CE to EA, so is the triangle CDE to the triangle ADE;

*therefore the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE;* (v. 11.)

that is, the triangles BDE, CDE, have the same ratio to the triangle ADE;

*therefore the triangle BDE is equal to the triangle CDE;* (v. 9.)

and they are on the same base DE;

but equal triangles on the same base and on the same side of it, are between the same parallels; (i. 39.)

**therefore DE is parallel to BC.**

Wherefore, if a straight line, &c.

Q. E. D.

### PROP. III.—THEOREM.

*If the angle of a triangle be divided into two equal angles, by a straight line which also cuts the base;*

*then the segments of the base shall have the same ratio which the other sides of the triangle have to one another;*

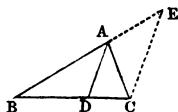
*and conversely, if the segments of the base have the same ratio which the other sides of the triangle have to one another;*

*then the straight line drawn from the vertex to the point of section, divides the vertical angle into two equal angles.*

(References—Prop. I. 5, 6, 29, 31; v. 7, 9, 11; vi. 2.)

Let ABC be a triangle, and let the angle BAC be divided into two equal angles by the straight line AD.

**Then shall BD be to DC, as BA to AC.**



### CONSTRUCTION

Through the point C draw CE parallel to DA; (i. 31.)  
and let BA produced meet CE in E.



## DEMONSTRATION

Because the straight line AC meets the parallels AD, EC,  
*therefore the angle ACE is equal to the alternate angle CAD ; (I. 29.)*  
 but CAD, by the hypothesis, is equal to the angle BAD ;  
*therefore BAD is equal to the angle ACE ; (ax. 1.)*  
 Again, because the straight line BAE meets the parallels AD, EC,  
*therefore the outward angle BAD is equal to the inward and opposite*  
*angle AEC ; (I. 29.)*  
 but the angle ACE has been proved equal to the angle BAD ;  
 therefore also ACE is equal to the angle AEC ; (ax. 1.)  
*wherefore the side AE is equal to the side AC ; (I. 6.)*  
 and because AD is drawn parallel to EC, one of the sides of the  
 triangle BCE,  
*therefore BD is to DC, as BA to AE ; (VI. 2.)*  
 but AE is equal to AC ;  
**therefore, as BD to DC, so is BA to AC. (v. 7.)**

*Next, let BD be to DC, as BA to AC, and join AD.*

**Then the angle BAC shall be divided into two equal angles by the straight line AD.**

## DEMONSTRATION

The same construction being made ;  
 because, as BD to DC, so is BA to AC ;  
 and as BD to DC, so is BA to AE, because AD is parallel to EC ;  
 (VI. 2.)  
*therefore BA is to AC, as BA to AE ; (v. 11.)*  
 therefore AC is equal to AE ; (v. 9.)  
*and therefore the angle AEC is equal to the angle ACE ; (I. 5.)*  
 but the angle AEC is equal to the outward and opposite angle BAD ;  
 and the angle ACE is equal to the alternate angle CAD ; (I. 29.)  
*wherefore also the angle BAD is equal to the angle CAD ; (ax. 1.)*  
**that is, the angle BAC is cut into two equal angles by**  
**the straight line AD.**

Therefore, if the angle, &c.

Q. E. D.

PROP. A.—THEOREM.

*If the outward angle of a triangle made by producing one of its sides, be divided into two equal angles, by a straight line, which also cuts the base produced;*

*then the segments between the dividing line and the extremities of the base, have the same ratio which the other sides of the triangle have to one another;*

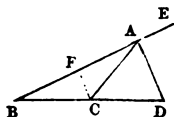
*and conversely, if the segments of the base produced have the same ratio which the other sides of the triangles have;*

*then the straight line drawn from the vertex to the point of section divides the outward angle of the triangle into two equal angles.*

(References—Prop. I. 5, 6, 29, 31; V. 7, 9, 11; VI. 2.)

Let ABC be a triangle, and let one of its sides BA be produced to E; and let the outward angle CAE be divided into two equal angles by the straight line AD which meets the base produced in D.

**Then BD shall be to DC, as BA to AC.**



CONSTRUCTION

Through C draw CF parallel to AD; (I. 31.)

DEMONSTRATION

And because the straight line AC meets the parallels AD, FC,

*the angle ACF is equal to the alternate angle CAD; (I. 29.)*

but CAD is equal to the angle DAE; (hyp.)

*therefore also DAE is equal to the angle ACF; (ax. 1.)*

Again, because the straight line FAE meets the parallels AD, FC,

*the outward angle DAE is equal to the inward and opposite angle CFA;*

*(I. 29.)*

but the angle ACF has been proved equal to the angle DAE;

*therefore also the angle ACF is equal to the angle CFA; (ax. 1.)*

*and therefore the side AF is equal to the side AC; (I. 6.)*

and because AD is parallel to FC, a side of the triangle BCF,

therefore  $BD$  is to  $DC$ , as  $BA$  to  $AF$  ; (vi. 2.)  
 but  $AF$  is equal to  $AC$  ;  
 therefore, as  $BD$  is to  $DC$ , so is  $BA$  to  $AC$ . (v. 7.)

Next, let  $BD$  be to  $DC$ , as  $BA$  to  $AC$ , and join  $AD$ .

Then the angle  $CAD$  shall be equal to the angle  $DAB$ .

#### DEMONSTRATION

The same construction being made,  
 because  $BD$  is to  $DC$ , as  $BA$  to  $AC$  ;  
 and that  $BD$  is also to  $DC$ , as  $BA$  to  $AF$  ; (vi. 2.)  
 therefore  $BA$  is to  $AC$ , as  $BA$  to  $AF$  ; (v. 11.)  
 wherefore  $AC$  is equal to  $AF$  ; (v. 9.)  
 and therefore the angle  $AFC$  equal to the angle  $ACF$  ; (l. 5.)  
 but the angle  $AFC$  is equal to the outward angle  $EAD$ , and the angle  
 $ACF$  to the alternate angle  $CAD$  ; (l. 29.)  
 therefore also  $EAD$  is equal to the angle  $CAD$ . (ax. 1.)  
 Wherefore, if the outward, &c. Q. E. D.

#### PROP. IV.—THEOREM.

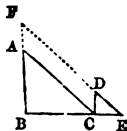
*The sides about the equal angles of equiangular triangles are proportionals ;*

*and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.*

(References—Prop. I. 17, 22, 28, 32, 34 ; v. 7, 16, 22 ; vi. 2.)

Let  $ABC$ ,  $DCE$  be equiangular triangles, having the angle  $ABC$  equal to the angle  $DCE$ , and the angle  $ACB$  to the angle  $DEC$  ; and, consequently, the angle  $BAC$  equal to the angle  $CDE$  ; (l. 32.)

Then the sides about the equal angles of the triangles  $ABC$ ,  $DCE$  shall be proportionals ; and those shall be the homologous sides which are opposite to the equal angles.



## CONSTRUCTION

Let the triangle DCE be placed, so that its side CE may be contiguous to BC, and in the same straight line with it ; (I. 22.)

because the angle BCA is equal to the angle CED, (hyp.)

add to each the angle ABC ;

*therefore the two angles ABC, BCA are equal to the two angles ABC, CED ; (ax. 2.)*

but the angles ABC, BCA are together less than two right angles ; (I. 17.)

*therefore the angles ABC, CED are also less than two right angles ;*

*wherefore BA, ED if produced will meet ; (I. ax. 12.)*

let them be produced and meet in the point F.

## DEMONSTRATION

Then because the angle ABC is equal to the angle DCE ; (hyp.)

*therefore BF is parallel to CD ; (I. 28.)*

and because the angle ACB is equal to the angle DEC,

*therefore AC is parallel to FE ; (I. 28.)*

*wherefore FA CD is a parallelogram ;*

*and consequently AF is equal to CD, and AC to FD ; (I. 34.)*

And because AC is parallel to FE, one of the sides of the triangle FBE,

*therefore BA is to AF, as BC to CE ; (vi. 2.)*

but AF is equal to CD ;

*therefore, as BA to CD, so is BC to CE ; (v. 7.)*

*and alternately, as AB to BC, so is DC to CE ; (v. 16.)*

Again, because CD is parallel to BF,

*therefore as BC to CE, so is FD to DE ; (vi. 2.)*

but FD is equal to AC ;

*therefore, as BC to CE, so is AC to DE ; (v. 7.)*

*and alternately, as BC to CA, so CE to ED ; (v. 16.)*

since then it has been shown that AB is to BC, as DC to CE, and as BC to CA, so CE to ED,

*therefore, ex æquali, BA is to AC, as CD to DE. (v. 22.)*

Therefore the sides, &c.

Q. E. D.

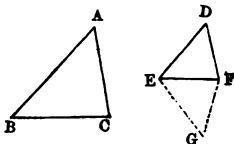
## PROP. V.—THEOREM.

*If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular; and the equal angles shall be those which are opposite to the homologous sides.*

(References—Prop. I. 4, 8, 23, 32 ; v. 9, 11 ; VI. 4.)

Let the triangles ABC, DEF have their side proportionals, so that AB is to BC, as DE to EF, and BC to CA, as EF to FD ; and consequently, ex æquali, BA to AC, as ED to DF.

**Then the triangle ABC shall be equiangular to the triangle DEF, and the angles which are opposite to the homologous sides shall be equal, viz. the angle ABC equal to the angle DEF, and BCA to EDF, and also BAC to EDF.**



## CONSTRUCTION

At the points E, F, in the straight line EF, make the angle FEG equal to the angle ABC, and the angle EFG equal to BCA ; (I. 23.)

## DEMONSTRATION

Because the remaining angle EGF, is equal to the remaining angle BAC ; (I. 32.)

*the triangle GEF is therefore equiangular to the triangle ABC ;*

*consequently they have their sides opposite to the equal angles proportionals ; (VI. 4.)*

*wherefore, as AB to BC, so is GE to EF ;*

but as AB to BC, so is DE to EF ; (hyp.)

*therefore as DE to EF, so GE to EF ; (v. 11.)*

*that is, DE and GE have the same ratio to EF, and consequently are equal. (v. 9.)*

For the same reason, DF is equal to FG.

And in the triangles DEF, GEF, because DE is equal to EG, and EF is common ;

the two sides DE, EF are equal to the two GE, EF, each to each ;

*and the base DF is equal to the base GF ;*

*therefore the angle DEF is equal to the angle GEF ; (I. 8.)*

and the other angles to the other angles which are subtended by the equal sides ; (I. 4.)

*therefore the angle DFE is equal to the angle GFE, and EDF to EGF ;*

and because the angle DEF is equal to the angle GEF,

and GEF equal to the angle ABC ; (constr.)

*therefore the angle ABC is equal to the angle DEF ; (ax. 1.)*

for the same reason, the angle ACB is equal to the angle DFE, and the angle at A equal to the angle at D ;

**therefore the triangle ABC is equiangular to the triangle DEF.**

Wherefore, if the sides, &c.

Q. E. D.

#### PROP. VI.—THEOREM.

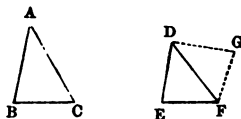
*If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals,*

*then the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.*

(References—Prop. I. 4, 23, 32 ; v. 9, 11 ; VI. 4.)

Let the triangles ABC, DEF have the angle BAC in the one equal to the angle EDF in the other, and the sides about those angles proportionals ; that is, BA to AC, as ED to DF.

**Then the triangles ABC, DEF shall be equiangular, and shall have the angle ABC equal to the angle DEF, and ACB to DFE.**



#### CONSTRUCTION

At the points D, F, in the straight line DF, make the angle FDG equal to either of the angles BAC, EDF, and the angle DFG equal to the angle ACB ; (I. 23.)

#### DEMONSTRATION

Because the remaining angle at B is equal to the remaining angle at G ; (I. 32.)

therefore the triangle DGF is equiangular to the triangle ABC ;  
*and therefore as BA to AC, so is GD to DF ; (vi. 4.)*  
 but, by the hypothesis, as BA to AC, so is ED to DF ;  
 therefore as ED to DF, so is GD to DF ; (v. 11.)  
 wherefore ED is equal to DG. (v. 9.)  
 And in the two triangles EDF, GDF ;  
 because ED is equal to DG, and DF common,  
 the two sides ED, DF are equal to the two sides GD, DF, each to each ;  
 and the angle EDF is equal to the angle GDF ; (constr.)  
*therefore the base EF is equal to the base FG, and the triangle EDF is equal to the triangle GDF ;*  
 and the remaining angles to the remaining angles, each to each, which are subtended by the equal sides,  
*therefore the angle DFG is equal to the angle DFE, and the angle at G to the angle at E ; (i. 4.)*  
 but the angle DFG is equal to the angle ACB ; (constr.)  
*therefore the angle ACB is equal to the angle DFE ; (ax. 1.)*  
 and the angle BAC is equal to the angle EDF ; (hyp.)  
*wherefore also the remaining angle at B is equal to the remaining angle at E ; (i. 32.)*  
**therefore the triangle ABC is equiangular to the triangle DEF.**

Wherefore, if two angles, &c.

Q. E. D.

#### PROP. VII.—THEOREM.

*If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals ;*

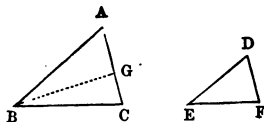
*then if each of the remaining angles be either less, or not less, than a right angle, or if one of them be a right angle ; the triangles shall be equiangular, and shall have those angles equal about which the sides are proportionals.*

(References—Prop. I. 5, 13, 17, 23, 32 ; v. 9, 11 ; vi. 4.)

Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other, viz. the angle BAC to the angle EDF, and the sides about two other angles ABC, DEF proportionals, so that AB is to BC, as DE to EF.

And *first*, let each of the remaining angles at C, F, be less than a right angle.

**Then the triangle ABC shall be equiangular to the triangle DEF, viz. the angle ABC shall be equal to the angle DEF, and the remaining angle at C equal to the remaining angle at F.**



## CONSTRUCTION

For if the angles ABC, DEF be not equal, one of them must be greater than the other;

let ABC be the greater, and at the point B, in the straight line AB, make the angle ABG equal to the angle DEF ; (I. 23.)

## DEMONSTRATION

Then because the angle at A is equal to the angle at D ; (hyp.)

and the angle ABG is assumed to be to the angle DEF ; (constr.)

therefore the remaining angle AGB must be equal to the remaining angle DFE ; (I. 32.)

*therefore the triangle ABG must be equiangular to the triangle DEF ;*

wherefore as AB is to BG, so must DE be to EF ; (VI. 4.)

but as DE to EF, so, by hypothesis, is AB to BC ;

*therefore as AB to BC, so must AB be to BG ; (v. 11.)*

and because AB has the same ratio to each of the lines BC, BG ;

therefore BC must be equal to BG ; (v. 9.)

*and therefore the angle BGC must be equal to the angle BCG ; (I. 5.)*

but the angle BCG is, by hypothesis, less than a right angle ;

therefore also the angle BGC must be less than a right angle ;

*and therefore the adjacent angle AGB must be greater than a right angle ; (I. 13.)*

but it was proved that the angle AGB 'must' be equal to the angle at F ;

*therefore the angle at F must be greater than a right angle.*

But, by the hypothesis, it is less than a right angle ; which is absurd ;

*therefore the angles ABC, DEF, are not unequal, that is, they are equal ;*

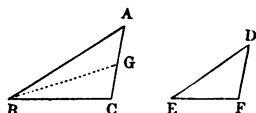


and the angle at A is equal to the angle at D ; (hyp.)

*wherefore the remaining angle at C is equal to the remaining angle at F ; (I. 32.)*

**therefore the triangle ABC is equiangular to the triangle DEF.**

*Secondly*, let each of the angles at C, F, be not less than a right angle. Then the triangle ABC shall be equiangular to the triangle DEF.



#### DEMONSTRATION

The same construction being made,

it may be proved in like manner that BC must be equal to BG,

*and therefore the angle at C equal to the angle BGC ; (I. 5.)*

but the angle at C is not less than a right angle ; (hyp.)

therefore the angle BGC cannot be less than a right angle ;

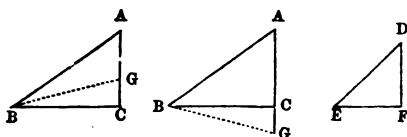
*wherefore two angles of the triangle BGC must be together not less than two right angles ;*

which is impossible ; (I. 17.)

**and therefore the triangle ABC may be proved to be equiangular to the triangle DEF, as in the first case.**

*Lastly*, let one of the angles at C, F, viz. the angle at C, be a right angle.

Then the triangle ABC shall be equiangular to the triangle DEF.



#### CONSTRUCTION

For if they be not equiangular, at the point B in the straight line AB make the angle ABG equal to the angle DEF.

#### DEMONSTRATION

It may be proved, as in the first case, that BG must be equal to BC;

*and therefore the angle BCG equal to the angle BGC ; (I. 5.)*

but the angle BCG is a right angle ; (hyp.)

therefore the angle BGC must be also a right angle ; (ax. 1.)

*wherefore two of the angles of the triangle BGC must be together not less than two right angles ;*

which is impossible ; (i. 17.)

**therefore the triangle ABC is equiangular to the triangle DEF.**

Wherefore, if two triangles, &c.

Q. E. D.

PROP. VIII.—THEOREM.

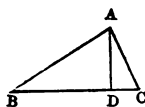
*In a right-angled triangle, if a perpendicular be drawn from the right angle to the base ;*

*then the triangles on each side of it are similar to the whole triangle, and to one another.*

(References—Prop. I. 32; VI. 4, def. 1.)

Let ABC be a right-angled triangle, having the right angle BAC ; and from the point A let AD be drawn perpendicular to the base BC.

**Then the triangles ABD, ADC shall be similar to the whole triangle ABC, and to one another.**



DEMONSTRATION

Then in the triangles ABC, ABD,

because the angle BAC is equal to the angle ADB, each of them being a right angle ; (ax. 11.)

and that the angle at B is common to both triangles ;

therefore the remaining angle ACB is equal to the remaining angle BAD ; (i. 32.)

*therefore the triangle ABC is equiangular to the triangle ABD, and the sides about their equal angles are proportionals ; (vi. 4.)*

**wherefore the triangle ABC is similar to the triangle ABD.** (vi. def. 1.)

In the same manner it may be demonstrated, that

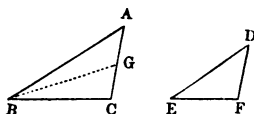
**the triangle ADC is similar to the triangle ABC.**

and the angle at A is equal to the angle at D ; (hyp.)

*wherefore the remaining angle at C is equal to the remaining angle at F ; (I. 32.)*

**therefore the triangle ABC is equiangular to the triangle DEF.**

*Secondly*, let each of the angles at C, F, be not less than a right angle. Then the triangle ABC shall be equiangular to the triangle DEF.



#### DEMONSTRATION

The same construction being made,

it may be proved in like manner that BC must be equal to BG,

*and therefore the angle at C equal to the angle BGC ; (I. 5.)*

but the angle at C is not less than a right angle ; (hyp.)

therefore the angle BGC cannot be less than a right angle ;

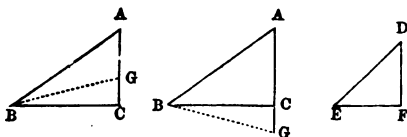
*wherefore two angles of the triangle BGC must be together not less than two right angles ;*

which is impossible ; (I. 17.)

**and therefore the triangle ABC may be proved to be equiangular to the triangle DEF, as in the first case.**

*Lastly*, let one of the angles at C, F, viz. the angle at C, be a right angle.

Then the triangle ABC shall be equiangular to the triangle DEF.



#### CONSTRUCTION

For if they be not equiangular, at the point B in the straight line AB make the angle ABG equal to the angle DEF.

#### DEMONSTRATION

It may be proved, as in the first case, that BG must be equal to BC;

*and therefore the angle BCG equal to the angle BGC ; (I. 5.)*

but the angle BCG is a right angle ; (hyp.)

therefore the angle BGC must be also a right angle ; (ax. 1.)

*wherefore two of the angles of the triangle BGC must be together not less than two right angles ;*

which is impossible ; (I. 17.)

**therefore the triangle ABC is equiangular to the triangle DEF.**

Wherefore, if two triangles, &c.

Q. E. D.

### PROP. VIII.—THEOREM.

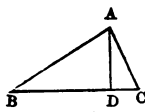
*In a right-angled triangle, if a perpendicular be drawn from the right angle to the base ;*

*then the triangles on each side of it are similar to the whole triangle, and to one another.*

(References—Prop. I. 32 ; VI. 4, def. 1.)

Let ABC be a right-angled triangle, having the right angle BAC ; and from the point A let AD be drawn perpendicular to the base BC.

**Then the triangles ABD, ADC shall be similar to the whole triangle ABC, and to one another.**



#### DEMONSTRATION

Then in the triangles ABC, ABD,

because the angle BAC is equal to the angle ADB, each of them being a right angle ; (ax. 11.)

and that the angle at B is common to both triangles ;

therefore the remaining angle ACB is equal to the remaining angle BAD ; (I. 32.)

*therefore the triangle ABC is equiangular to the triangle ABD, and the sides about their equal angles are proportionals ; (VI. 4.)*

**wherefore the triangle ABC is similar to the triangle ABD.** (VI. def. 1.)

In the same manner it may be demonstrated, that

**the triangle ADC is similar to the triangle ABC.**

And because the triangles ABD, ACD, are both similar to ABC,

**therefore they are similar to each other.**

Therefore, in a right-angled, &c.

Q. E. D.

Cor. From this it is manifest, that the perpendicular drawn from the right angle of a right-angled triangle to the base is a mean proportional between the segments of the base ; and also that each of the sides is a mean proportional between the base and the segment of it adjacent to that side ;

For, in the triangles BDA, ADC,

BD is to DA, as DA to DC ; (VI. 4.)

and in the triangles ABC, DBA,

BC is to BA, as BA to BD ; (VI. 4.)

and in the triangles ABC, ACD,

BC is to CA, as CA to CD ; (VI. 4.)

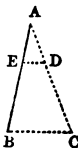
### PROP. IX.—PROBLEM.

*From a given straight line to cut off any part required.*

(References—Prop. I. 31 ; v. D, 18 ; VI. 2.)

Let AB be the given straight line.

It is required to cut off any part from it.



### CONSTRUCTION

From the point A draw a straight line AC, making any angle with AB, and in AC take any point D ;

and take AC the same multiple of AD, that AB is of the part which is to be cut off from it ;

join BC, and draw DE parallel to CB. (I. 31.)

**Then AE shall be the part required to be cut off.**

## DEMONSTRATION

Because  $ED$  is parallel to  $BC$ , one of the sides of the triangle  $ABC$ ,  
*therefore as  $CD$  is to  $DA$ , so is  $BE$  to  $EA$  ; (VI. 2.)*

and by composition,

*CA is to AD, as BA to AE ; (v. 18.)*

but  $CA$  is a multiple of  $AD$  ; (constr.)

*therefore BA is the same multiple of AE ; (v. D.)*

that is, whatever part  $AD$  is of  $AC$ ,  $AE$  is the same part of  $AB$  ;

**wherefore, from the straight line  $AB$  the part required  
 is cut off.**

Q. E. F.

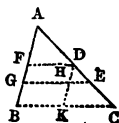
## PROP. X.—PROBLEM.

*To divide a given straight line similarly to a given divided straight line, that is, into parts that shall have the same ratios to one another, which the parts of the divided given straight line have.*

(References—Prop. I. 31, 34 ; v. 7 ; VI. 2.)

Let  $AB$  be the straight line given to be divided, and  $AC$  the divided line.

It is required to divide  $AB$  similarly to  $AC$ .



## CONSTRUCTION

Let  $AC$  be divided in the points  $D, E$  ;

and let  $AB, AC$  be placed so as to contain any angle, and join  $BC$ ,  
 and through the points  $D, E$  draw  $DF, EG$  parallels to  $BC$  ; (I. 31.)

**Then  $AB$  shall be divided in the points  $F, G$ , similarly to  
 $AC$ .**

Through  $D$  draw  $DHK$  parallel to  $AB$ .

## DEMONSTRATION

Because each of the figures  $FH, HB$  is a parallelogram ; (constr.)

*therefore  $DH$  is equal to  $FG$ , and  $HK$  to  $GB$  ; (I. 34.)*

and because HE is parallel to KC, one of the sides of the triangle DKC,  
*therefore, as CE to ED, so is KH to HD ; (vl. 2.)*

but KH is equal to BG, and HD to GF ;

*therefore, as CE is to ED, so is BG to GF ; (v. 7.)*

Again, because FD is parallel to GE, one of the sides of the triangle AGE,

*therefore, as ED is to DA, so is GF to FA ; (vl. 2.)*

and it has been shown that as CE is to ED, so is BG to GF, and as ED is to DA, so is GF to FA ;

**therefore the given straight line AB is divided similarly to AC.**  
 Q. E. F.

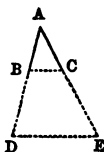
### PROP. XI.—PROBLEM.

*To find a third proportional to two given straight lines.*

(References—Prop. I. 3, 31 ; v. 7 ; vl. 2.)

Let AB, AC be the two given straight lines.

It is required to find a third proportional to AB, AC.



### CONSTRUCTION

Let AB, AC be placed so as to contain any angle ;

produce AB, AC to the points D, E, and make BD equal to AC ; (I. 3.)

join BC, and through D, draw DE parallel to BC. (I. 31.)

**Then CE shall be a third proportional to AB and AC.**

### DEMONSTRATION

Because BC is parallel to DE, a side of the triangle ADE,

*therefore, AB is to BD, as AC to CE ; (vl. 2.)*

but BD is equal to AC ;

*therefore, as AB is to AC, so is AC to CE ; (v. 7.)*

**wherefore, to the two given straight lines AB, AC, a third proportional CE is found.**  
 Q. E. F.

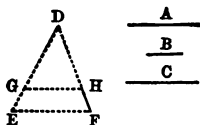
## PROP. XII.—PROBLEM.

*To find a fourth proportional to three given straight lines.*

(References—Prop. I. 3, 31 ; v. 7 ; VI. 2.)

Let A, B, C be the three given straight lines.

It is required to find a fourth proportional to A, B, C.



## CONSTRUCTION

Take two straight lines DE, DF, containing any angle EDF; and upon these make DG equal to A, GE equal to B, and DH equal to C; (I. 3.) join GH, and through E draw EF parallel to it. (I. 31.)

**Then HF shall be the fourth proportional to A, B, C.**

## DEMONSTRATION

Because GH is parallel to EF, one of the sides of the triangle DEF, therefore DG is to GE, as DH to HF; (VI. 2.)

but DG is equal to A, GE to B, and DH to C; (constr.)

therefore, as A is to B, so is C to HF; (v. 7.)

**wherefore to the three given straight lines A, B, C, a fourth proportional HF is found.**

Q. E. F.

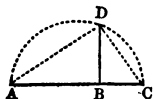
## PROP. XIII.—PROBLEM.

*To find a mean proportional between two given straight lines.*

(References—Prop. I. 11 ; III. 31; VI. 8 cor.)

Let AB, BC be the two given straight lines.

It is required to find a mean proportional between them.





## CONSTRUCTION

Place AB, BC in a straight line, and upon AC describe the semicircle ADC,

and from the point B draw BD at right angles to AC. (I. 11.)

**Then BD shall be a mean proportional between AB and BC.**

Join AD, DC.

## DEMONSTRATION

Because the angle ADC in a semicircle is a right angle ; (III. 31.)  
and because in the right-angled triangle ADC, BD is drawn from the  
right angle perpendicular to the base,

*therefore DB is a mean proportional between AB, BC, the segments of  
the base ; (VI. 8 cor.)*

**therefore between the two given straight lines AB, BC,  
a mean proportional DB is found.**

Q. E. F.

## PROP. XIV.—THEOREM.

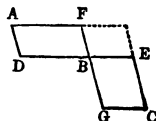
*Equal parallelograms, which have one angle of the one equal to one  
angle of the other, have their sides about the equal angles reciprocally  
proportional ;*

*and conversely, parallelograms that have one angle of the one equal to  
one angle of the other, and their sides about the equal angles reciprocally  
proportional, are equal to one another.*

(References—Prop. I. 14 ; v. 7, 9, 11 ; VI. 1.)

Let AB, BC be equal parallelograms, which have the angles at B  
equal.

**Then the sides of the parallelograms AB, BC about the  
equal angles shall be reciprocally proportional ; that  
is, DB shall be to BE, as GB to BF.**



## CONSTRUCTION

Let the sides DB, BE be placed in the same straight line;  
 therefore also FB, BG are in one straight line ; (I. 14.)  
 complete the parallelogram FE.

## DEMONSTRATION

And because the parallelogram AB is equal to BC, and that FE is another parallelogram,

*therefore AB is to FE, as BC to FE ; (v. 7.)*

but as AB to FE, so is the base DB to BE ; (vi. 1.)

and as BC to FE, so is the base GB to BF ;

*therefore, as DB to BE, so is GB to BF ; (v. 11.)*

**wherefore, the sides of the parallelograms AB, BC about their equal angles are reciprocally proportional.**

*Next*, let the sides of the parallelograms AB, BC about their angles be reciprocally proportional, viz. as DB to BE, so GB to BF.

**Then the parallelogram AB shall be equal to the parallelogram BC.**

## DEMONSTRATION

The same construction being made,

because, as DB to BE, so is GB to BF;

and as DB to BE, so is the parallelogram AB to the parallelogram FE ; (vi. 1.)

and as GB to BF, so is the parallelogram BC to the parallelogram FE ;

*therefore as AB to FE, so is BC to FE ; (v. 11.)*

**therefore the parallelogram AB is equal to the parallelogram BC. (v. 9.)**

Therefore equal parallelograms, &c.

Q. E. D.

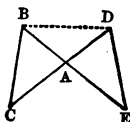
## PROP. XV.—THEOREM.

*Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional ; and conversely, triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.*

(References—Prop. I. 14 ; v. 7, 9, 11 ; VI. 1.)

Let  $ABC$ ,  $ADE$  be equal triangles, which have the angle  $BAC$  equal to the angle  $DAE$ .

**Then the sides about the equal angles of the triangles shall be reciprocally proportional; that is,  $CA$  shall be to  $AD$ , as  $EA$  to  $AB$ .**



#### CONSTRUCTION

Let the triangles be placed so that their sides  $CA$ ,  $AD$  be in one straight line ;

therefore also  $EA$  and  $AB$  are in one straight line; (I. 14.)

and join  $BD$ .

#### DEMONSTRATION

Because the triangle  $ABC$  is equal to the triangle  $ADE$ , and that  $ABD$  is another triangle ;

*therefore as the triangle  $CAB$  is to the triangle  $BAD$ , so is the triangle  $AED$  to the triangle  $DAB$  ; (v. 7.)*

but as the triangle  $CAB$  to the triangle  $BAD$ , so is the base  $CA$  to the base  $AD$  ; (vi. 1.)

and as the triangle  $EAD$  to the triangle  $DAB$ , so is the base  $EA$  to the base  $AB$  ; (vi. 1.)

*therefore as  $CA$  to  $AD$ , so is  $EA$  to  $AB$  ; (v. 11.)*

**therefore the sides of the triangles  $ABC$ ,  $ADE$  about the equal angles are reciprocally proportional.**

*Next*, let the sides of the triangles  $ABC$ ,  $ADE$  about the equal angles be reciprocally proportional, viz.  $CA$  to  $AD$  as  $EA$  to  $AB$ .

**Then the triangle  $ABC$  shall be equal to the triangle  $ADE$ .**

#### DEMONSTRATION

The same construction being made,

because, as  $CA$  to  $AD$ , so is  $EA$  to  $AB$  ; (hyp.)

and as  $CA$  to  $AD$ , so is the triangle  $ABC$  to the triangle  $BAD$  ; (vi. 1.)

and as EA to AB, so is the triangle EAD to the triangle BAD ;  
(vi. 1.)

*therefore as the triangle BAC to the triangle BAD, so is the triangle EAD to the triangle BAD ; (v. 11.)*

that is, the triangles BAC, EAD have the same ratio to the triangle BAD ;

**wherefore the triangle ABC is equal to the triangle ADE.**  
(v. 9.)

Therefore, equal triangles, &c.

Q. E. D.

### PROP. XVI.—THEOREM.

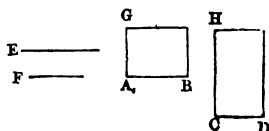
*If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means ;*

*and conversely, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.*

(References—Prop. I. 3, 11, 31 ; v. 7 ; vi. 14.)

Let the four straight lines AB, CD, E, F be proportionals, viz. as AB to CD, so E to F.

**Then the rectangle contained by AB, and F shall be equal to the rectangle contained by CD, and E.**



#### CONSTRUCTION

From the points A, C draw AG, CH at right angles to AB, CD ;  
(I. 11.)

make AG equal to F, and CH equal to E ; (I. 3.)

and complete the parallelograms BG, DH. (I. 31.)

#### DEMONSTRATION

Because, as AB to CD, so is E to F ;

and that E is equal to CH, and F to AG ;

therefore AB is to CD as CH to AG ; (v. 7.)

*that is, the sides of the parallelograms BG, DH about the equal angles are reciprocally proportional ;*

but parallelograms which have their sides about equal angles reciprocally proportional, are equal to one another ; (VI. 14.)

*therefore the parallelogram BG is equal to the parallelogram DH.*

But the parallelogram BG is contained by the straight lines AB and F, because AG is equal to F ;

and the parallelogram DH is contained by CD, E, because CH is equal to E ;

**therefore the rectangle contained by the straight lines AB, F, is equal to that which is contained by CD and E.**

*Next, if the rectangle contained by the straight lines AB, F, be equal to that which is contained by CD, E.*

**These four lines shall be proportional, viz. AB shall be to CD, as E to F.**

#### DEMONSTRATION

The same construction being made,

because the rectangle contained by the straight lines AB, and F, is equal to that which is contained by CD, and E,

and that the rectangle BG is contained by AB, and F, because AG is equal to F ;

and the rectangle DH by CD, and E, because CH is equal to E ;

*therefore the parallelogram BG is equal to the parallelogram DH ;*  
(ax. 1.)

and they are equiangular.

But their sides about the equal angles of equal parallelograms are reciprocally proportional ; (VI. 14.)

*wherefore, as AB to CD, so is CH to AG ;*

but CH is equal to E, and AG to F ;

**therefore as AB to CD, so is E to F. (V. 7.)**

Wherefore, if four, &c.

Q. E. D.

#### PROP. XVII.—THEOREM.

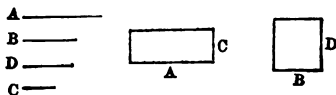
*If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square on the mean ;*

*and conversely, if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines are proportionals.*

(References—Prop. V. 7 ; VI. 16.)

Let the three straight lines A, B, C be proportionals, viz. as A to B, so B to C.

**Then the rectangle contained by A, C, shall be equal to the square on B.**



## CONSTRUCTION

Take D equal to B.

## DEMONSTRATION

Because as A to B, so B to C, and that B is equal to D ;

*therefore as A is to B, so D to C ; (v. 7.)*

but if four straight lines be proportionals, the rectangle contained by the extremes is equal to that which is contained by the means ; (vi. 16.)

*therefore the rectangle contained by A, C, is equal to that contained by B, D ;*

but the rectangle contained by B, D is the square on B, because B is equal to D ;

**therefore the rectangle contained by A, C, is equal to the square on B.**

*Next, if the rectangle contained by A, C, be equal to the square on B.*

**Then A shall be to B, as B to C.**

## DEMONSTRATION

The same construction being made,

because the rectangle contained by A, C, is equal to the square on B, and the square on B is equal to the rectangle contained by B, D, because B is equal to D ;

*therefore the rectangle contained by A, and C, is equal to that contained by B, and D.*

But if the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportionals ; (vi. 16.)

*therefore A is to B, as D to C ;*

but B is equal to D ;

**wherefore, as A to B, so B to C.**

Therefore, if three straight lines, &c.

**Q. E. D.**

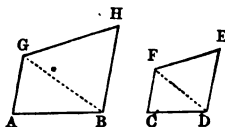
## PROP. XVIII.—PROBLEM.

*Upon a given straight line to describe a rectilineal figure similar, and similarly situated, to a given rectilineal figure.*

(References—Prop. I. 23, 32 ; v. 22 ; VI. 4.)

*First*, let  $AB$  be the given straight line, and  $CDEF$  the given rectilineal figure of four sides.

It is required upon the given straight line  $AB$  to describe a rectilineal figure similar, and similarly situated, to  $CDEF$ .



## CONSTRUCTION

Join  $DF$ , and at the points  $A, B$ , in the straight line  $AB$  make the angle  $BAG$  equal to the angle at  $C$ , and the angle  $ABG$  equal to the angle  $CDE$  ; (I. 23.)

and at the points  $G, B$ , in the straight line  $GB$ , make the angle  $BGH$  equal to the angle  $DFE$ , and the angle  $GBH$  equal to the angle  $FDE$ . (I. 23.)

**Then the figure  $ABHG$  shall be similar and similarly situated to  $CDEF$ .**

## DEMONSTRATION

Because the angle  $BAG$  is equal to  $C$ , and the angle  $ABG$  to the angle  $CDE$  ; (constr.)

therefore the remaining angle  $AGB$  is equal to the remaining angle  $CFD$  ; (I. 32.)

*therefore the triangle  $FCD$  is equiangular to the triangle  $GAB$ .*

and because the angle  $BGH$  is equal to the angle  $DFE$ , and the angle  $GBH$  to the angle  $FDE$  ; (constr.)

therefore the remaining angle  $GHB$  is equal to the remaining angle  $FED$ ,

*and the triangle  $FDE$  equiangular to the triangle  $GBH$ .*

Then, because the angle  $AGB$  is equal to the angle  $CFD$ , and  $BGH$  to  $DFE$ ,

*therefore the whole angle  $AGH$  is equal to the whole  $CFE$  ; (ax. 2.)*  
for the same reason,

*the angle ABH is equal to the angle CDE ;*  
 and the angle at A is equal to the angle at C ; (constr.)  
 and the angle GHB to FED ;  
*therefore the rectilineal figure ABHG is equiangular to CDEF.*

Likewise these figures have their sides about the equal angles proportionals ;

for, because the triangles GAB, FCD are equiangular,

*therefore BA is to AG, as CD to CF ; (vi. 4.)*

and because AG is to GB, as CF to FD, and as GB is to GH, so is FD to FE, by reason of the equiangular triangles BGH, DFE,

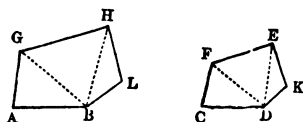
*therefore, ex æquali, AG is to GH, as CF to FE ; (v. 22.)*

In the same manner it may be proved that AB is to BH, as CD to DE :  
 and GH is to HB, as FE to ED ; (vi. 4.)

*therefore the rectilineal figures ABHG, CDEF are equiangular, and have their sides about the equal angles proportionals ;*

**wherefore ABHG is similar to CDEF. (vi. def. 1.)**

*Secondly*, let it be required to describe upon a given straight line AB, a rectilineal figure similar, and similarly situated, to the rectilineal figure CDKEF of five sides.



#### CONSTRUCTION

Join DE, and upon the given straight line AB describe the rectilineal figure ABHG similar, and similarly situated, to the quadrilateral figure CDEF, by the former case ;

and at the points B, H, in the straight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK.

**Then the figure ABLEH is similar, and similarly situated, to CDEF.**

#### DEMONSTRATION

Because the angle HBL is equal to the angle EDK, and the angle BHL equal to the angle DEK ; (constr.)



*therefore the remaining angle at L is equal to the remaining angle at K ;*  
(1. 32.)

and, because the figures ABHG, CDEF, are similar,  
*the angle GHB is equal to the angle FED ;* (vi. def. 1.)  
and BHL is equal to DEK ; (constr.)

*therefore the whole angle GHL is equal to the whole angle FEK ;*  
for the same reason, the angle ABL is equal to the angle CDK ;  
*therefore the five-sided figures AGHLB, CFEKD, are equiangular.*

And because the figures AGHB, CFED, are similar,  
*therefore GH is to HB, as FE to ED ;* (vi. def. 1.)  
but as HB to HL, so is ED to EK ; (vi. 4.)

*therefore, ex æquali, GH is to HL, as FE to EK ;* (v. 22.)  
for the same reason, AB is to BL, as CD to DK, and BL is to LH, as  
DK to KE, because the triangles BLH, DKE, are equiangular ;  
(vi. 4.)

*therefore, the five-sided figures ABLHG, CDKEF, are equiangular,*  
*and have their sides about the equal angles proportionals ;*

**wherefore ABLHG is similar to CDKEF.**

In the same manner a rectilineal figure of six sides [may be described  
upon a given straight line similar to one given, and so on.

Q. E. F.

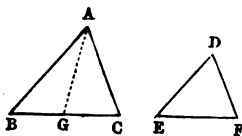
#### PROP. XIX.—THEOREM.

*Similar triangles are to one another in the duplicate ratio of their homologous sides.*

(References—Prop. v. 11, 16, def. 12; vi. 1, 11, 15.)

Let ABC, DEF be similar triangles, having the angle at B equal to the angle E, and let AB be to BC, as DE to EF, so that the side BC may be homologous to EF ; (v. def. 12.)

**Then the triangle ABC shall have to the triangle DEF the duplicate ratio of that which BC has to EF.**



## CONSTRUCTION

Take BG a third proportional to BC, EF, so that BC may be to EF,  
as EF to BG ; (VI. 11.)  
and join GA.

## DEMONSTRATION

Then, because, as AB to BC, so is DE to EF ; (hyp.)  
*therefore, alternately, AB is to DE, as BC to EF ; (v. 16.)*  
but as BC to EF, so is EF to BG ; (constr.)  
*therefore, as AB to DE, so is EF to BG ; (v. 11.)*  
*therefore the sides of the triangles ABG, DEF, which are about the*  
*equal angles, are reciprocally proportional ;*  
but triangles, which have the sides about two equal angles reciprocally  
proportional, are equal to one another ; (VI. 15.)  
*therefore the triangle ABG is equal to the triangle DEF.*  
And because as BC is to EF, so is EF to BG ;  
and that if three straight lines be proportionals, the first is said to have  
to the third, the duplicate ratio of that which it has to the second ;  
(v. def. 10.)  
*therefore BC has to BG the duplicate ratio of that which BC has to*  
*EF ;*  
but as BC is to BG, so is the triangle ABC to the triangle ABG ;  
(VI. 1.)  
*therefore the triangle ABC has to the triangle ABG, the duplicate ratio*  
*of that which BC has to EF ;*  
but the triangle ABG is equal to the triangle DEF ;  
**therefore also the triangle ABC has to the triangle DEF,**  
**the duplicate ratio of that which BC has to EF.**  
Therefore, similar triangles, &c. Q. E. D.

Cor. From this it is manifest, that if three straight lines be propor-  
tionals, as the first is to the third, so is any triangle upon the first to a  
similar and similarly described triangle upon the second.

## PROP. XX.—THEOREM.

*Similar polygons may be divided into the same number of similar*  
*triangles, having the same ratio to one another that the polygons have ; and*

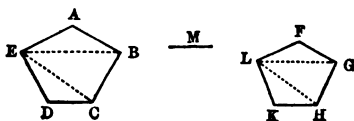
*the polygons have to one another the duplicate ratio of that which their homologous sides have.*

(References—Prop. I. 32; V. 11, 12, 22; VI. 4, 6, 11, 19, def. 1.)

Let  $ABCDE$ ,  $FGHKL$ , be similar polygons, and let  $AB$  be the side homologous to  $FG$ .

**Then the polygons  $ABCDE$ ,  $FGHKL$ , may be divided into the same number of similar triangles, whereof each shall have to each the same ratio which the polygons have;**

**and the polygon  $ABCDE$  shall have to the polygon  $FGHKL$  the duplicate ratio of that which the side  $AB$  has to the side  $FG$ .**



#### CONSTRUCTION

Join  $BE$ ,  $EC$ ,  $GL$ ,  $LH$ .

#### DEMONSTRATION

Because the polygon  $ABCDE$  is similar to the polygon  $FGHKL$ ,

*the angle  $BAE$  is equal to the angle  $GFL$ , and  $BA$  is to  $AE$ , as  $GF$  to  $FL$ ; (VI. def. 1.)*

and because the triangles  $ABE$ ,  $FGL$  have an angle in one, equal to an angle in the other, and their sides about these equal angles proportionals,

*therefore the triangle  $ABE$  is equiangular to the triangle  $FGL$ ; (VI. 6.) and therefore similar to it; (VI. 4.)*

*wherefore the angle  $ABE$  is equal to the angle  $FGL$ ;*

and, because the polygons are similar,

*the whole angle  $ABC$  is equal to the whole angle  $FGH$ ; (VI. def. 1.)*

*therefore the remaining angle  $EBC$  is equal to the remaining angle  $LGH$ . (I. 32 and ax. 3.)*

And because the triangles  $ABE$ ,  $FGL$  are similar,

*therefore  $EB$  is to  $BA$ , as  $LG$  to  $GF$ ; (VI. 4.)*

*and also, because the polygons are similar,*

therefore AB is to BC, as FG to GH ; (vi. def. 1.)

*therefore, ex æquali*, EB is to BC, as LG to GH ; (v. 22.)

that is, the sides about the equal angles EBC, LGH are proportionals ;

*therefore, the triangle EBC is equiangular to the triangle LGH*, (vi. 6.)

and similar to it ; (vi. 4.)

For the same reason, the triangle ECD likewise is similar to the triangle LHK ;

**therefore the similar polygons ABCDE, FGHLK are divided into the same number of similar triangles.**

Also these triangles shall have, each to each, the same ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK ; and the polygon ABCDE shall have to the polygon FGHLK the duplicate ratio of that which the side AB has to the homologous side FG.

Because the triangle ABE is similar to the triangle FGL,

*therefore ABE has to FGL, the duplicate ratio of that which the side BE has to the side GL ;* (vi. 19.)

for the same reason, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL ;

*therefore, as the triangle ABE is to the triangle FGL, so is the triangle BEC to the triangle GLH ;* (v. 11.)

Again, because the triangle EBC is similar to the triangle LGH,

*therefore EBC has to LGH, the duplicate ratio of that which the side EC has to the side LH ;*

for the same reason, the triangle ECD has to the triangle LHK, the duplicate ratio of that which EC has to LH ;

*therefore, as the triangle EBC is to the triangle LGH, so is the triangle ECD to the triangle LHK.* (v. 11.)

But it has been proved, that the triangle EBC is likewise to the triangle LGH, as the triangle ABE to the triangle FGL ;

*therefore, as the triangle ABE to the triangle FGL, so is the triangle EBC to the triangle LGH, and the triangle ECD to the triangle LHK ;*

and therefore, as one of the antecedents is to the consequents, so are all the antecedents to all the consequents ; (v. 12.)

*that is, as the triangle ABE to the triangle FGL, so is the polygon ABCDE to the polygon FGHLK ;*

but the triangle ABE has to the triangle FGL, the duplicate ratio of that which the side AB has to the homologous side FG ; (vi. 19.)

therefore also the polygon **ABCD** has to the polygon **FGHI** the duplicate ratio of that which **AB** has to the homologous side **FG**.

Wherefore, similar polygons, &c.

Q. E. D.

Cor. 1. In like manner it may be proved, that similar four-sided figures, or of any number of sides, are one to another in the duplicate ratio of their homologous sides; and it has already been proved in triangles; (vi. 19.) therefore, universally, similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

Cor. 2. And if to **AB**, **FG**, two of the homologous sides, a third proportional **M** be taken; (vi. 11.)

then **AB** has to **M** the duplicate ratio of that which **AB** has to **FG**; (v. def. 10.)

but the four-sided figure or polygon upon **AB**, has to the four-sided figure or polygon upon **FG**, likewise the duplicate ratio of that which **AB** has to **FG**; (vi. 20, cor. 1.)

therefore as **AB** is to **M**, so is the figure upon **AB** to the figure upon **FG**; (v. 11.)

which was also proved in triangles; (vi. 19, cor.)

therefore, universally, it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first, to a similar and similarly described rectilineal figure upon the second.

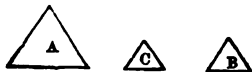
#### PROP. XXI.—THEOREM.

*Rectilineal figures which are similar to the same rectilineal figure, are also similar to one another.*

(References—Prop. v. 11; vi. def. 1.)

Let each of the rectilineal figures **A**, **B**, be similar to the rectilineal figure **C**.

Then the figure **A** shall be similar to the figure **B**.



#### DEMONSTRATION

Because **A** is similar to **C**,

they are equiangular, and also have their sides about the equal angles proportionals; (vi. def. 1.)

again, because B is similar to C,

*they are equiangular, and have their sides about the equal angles proportionals ; (VI. def. 1.)*

therefore the figures A, B, are each of them equiangular to C, and have the sides about the equal angles of each of them and of C proportionals ;

*wherefore the rectilinear figures A and B are equiangular, and have their sides about the equal angles proportionals ; (v. 11.)*

**therefore A is similar to B.** (VI. def. 1.)

Therefore, rectilinear figures, &c.

Q. E. D.

### PROP. XXII.—THEOREM.

*If four straight lines be proportionals, the similar rectilinear figures similarly described upon them shall also be proportionals ;*

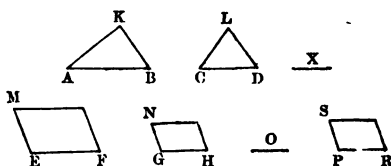
*and conversely, if the similar rectilinear figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.*

(References—Prop. v. 7, 9, 11, 22 ; VI. 11, 12, 18, 20, cor. 2.)

Let the four straight lines AB, CD, EF, GH, be proportionals, viz. AB to CD, as EF to GH ;

and upon AB, CD, let the similar rectilinear figures KAB, LCD, be similarly described ; and upon EF, GH, the similar rectilinear figures MF, NH, in like manner.

**Then the rectilinear figure KAB shall be to LCD, as MF to NH.**



### CONSTRUCTION

To AB, CD take a third proportional X, and to EF, GH, a third proportional O ; (VI. 11.)

### DEMONSTRATION

Because AB is to CD as EF to GH,  
therefore CD is to X, as GH to O ; (v. 11.)

*therefore, ex æquali, as AB to X, so EF to O; (v. 22.)*

but as AB to X, so is the rectilineal figure KAB to the rectilineal figure LCD,

and as EF to O, so is the rectilineal figure MF to the rectilineal figure NH; (vi. 20, cor. 2.)

**therefore, as KAB to LCD, so is MF to NH. (v. 11.)**

*Next, let the rectilineal figure KAB be to LCD, as MF to NH.*

**Then the straight line AB shall be to CD, as EF to GH.**

#### CONSTRUCTION

Make as AB to CD, so EF to PR; (vi. 12.)

and upon PR describe the rectilineal figure SR similar and similarly situated to either of the figures MF, NH; (vi. 18.)

#### DEMONSTRATION

Then because as AB to CD, so is EF to PR,

and that upon AB, CD are described the similar and similarly situated rectilineals KAB, LCD,

and upon EF, PR, in like manner, the similar rectilineals MF, SR;

*therefore KAB is to LCD, as MF to SR;*

but, by the hypothesis, KAB is to LCD, as MF to NH;

therefore the rectilineal MF has the same ratio to each of the two NH, SR.

*therefore SR is equal to NH; (v. 9.)*

But they are also similar, and similarly situated; (constr.)

*therefore GH is equal to PR;*

and because as AB is to CD, so is EF to PR,

and that PR is equal to GH;

**therefore AB is to CD, as EF to GH. (v. 7.)**

If, therefore, four straight lines, &c.

Q. E. D.

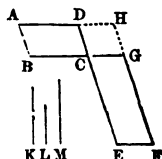
#### PROP. XXIII.—THEOREM.

*Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.*

(References—Prop. i. 14; v. 11, 22, def. A; vi. 1, 12.)

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG.

**Then the ratio of the parallelogram AC to the parallelogram CF, shall be the same with the ratio which is compounded of the ratios of their sides.**



## CONSTRUCTION

Let BC, CG be placed in a straight line,  
 then DC and CE shall also be in a straight line ; (I. 14.)  
 complete the parallelogram DG ;  
 and taking any straight line K, make as BC to CG, so K to L, and as  
 DC to CE, so make L to M ; (VI. 12.)

## DEMONSTRATION

Because the ratios of K to L, and L to M, are the same with the ratios of the sides, viz. of BC to CG, and DC to CE ; (constr.)

but the ratio of K to M is that which is said to be compounded of the ratios of K to L, and L to M ; (v. def. A.)

*therefore K has to M the ratio compounded of the ratios of the sides ;*

and because as BC to CG, so is the parallelogram AC to the parallelogram CH ; (vi. 1.)

but as BC to CG, so is K to L ;

*therefore K is to L, as the parallelogram AC to the parallelogram CH ;*  
 (v. 11.)

Again, because as DC to CE, so is the parallelogram CH to the parallelogram CF ;

but as DC to CE, so is L to M ;

*therefore L is to M, as the parallelogram CH to the parallelogram CF ;*  
 (v. 11.)

then, since it has been proved, that as K to L, so is the parallelogram AC to the parallelogram CH ;

and as L to M, so is the parallelogram CH to the parallelogram CF ;

*therefore, ex æquali, K is to M, as the parallelogram AC to the parallelogram CF ; (v. 22.)*



but  $K$  has to  $M$  the ratio which is compounded of the ratios of the sides ;

**therefore also the parallelogram  $AC$  has to the parallelogram  $CF$ , the ratio which is compounded of the ratios of the sides.**

Wherefore, equiangular parallelograms, &c.

Q. E. D.

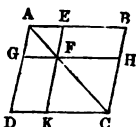
**PROP. XXIV.—THEOREM.**

*Parallelograms about the diameter of any parallelogram, are similar to the whole, and to one another.*

(References—Prop. I. 29, 34; v. 7; VI. 4, 21, def. 1.)

Let  $ABCD$  be a parallelogram, of which the diameter is  $AC$ ; and  $EG$ ,  $HK$  parallelograms about the diameter.

**Then the parallelograms  $EG$ ,  $HK$  shall be similar both to the whole parallelogram  $ABCD$ , and to one another.**



**DEMONSTRATION**

Because  $DC$ ,  $GF$  are parallels,

*the angle  $ADC$  is equal to the angle  $AGF$  ; (I. 29.)*

for the same reason, because  $BC$ ,  $EF$  are parallels,

*the angle  $ABC$  is equal to the angle  $AEF$  ;*

and each of the angles  $BCD$ ,  $EFG$  is equal to the opposite angle  $DAB$  ;  
(I. 34.)

therefore they are equal to one another ;

*wherefore the parallelograms  $ABCD$ ,  $AEFG$  are equiangular.*

And in the two triangles  $BAC$ ,  $EAF$ ,

because the angle  $ABC$  is equal to the angle  $AEF$ , and the angle  $BAC$  common to both ;

they are equiangular to one another ;

*therefore as  $AB$  is to  $BC$ , so is  $AE$  to  $EF$  ; (VI. 4.)*

and because the opposite sides of parallelograms are equal to one another ; (I. 34.)

therefore  $AB$  is to  $AD$  as  $AE$  to  $AG$ ; (v. 7.)

and  $DC$  to  $CB$ , as  $GF$  to  $FE$ ;

and also  $CD$  to  $DA$ , as  $FG$  to  $GA$ ;

*therefore the sides of the parallelograms  $ABCD$ ,  $AEEG$  about the equal angles are proportionals.*

and they are similar to one another; (vi. def. 1.)

For the same reason, the parallelogram  $ABCD$  is similar to the parallelogram  $FHCK$ ;

*wherefore each of the parallelograms  $GE$ ,  $KH$  is similar to  $DB$ ;*

but rectilineal figures which are similar to the same rectilineal figure, are also similar to one another; (vi. 21.)

**therefore the parallelogram  $GE$  is similar to  $KH$ .**

Wherefore, parallelograms, &c.

Q. E. D.

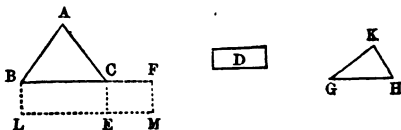
# PROP. XXV.—PROBLEM.

*To describe a rectilineal figure which shall be similar to one, and equal to another given rectilineal figure.*

(References—Prop. i. 14, 29, 45, cor. 1; v. 11, 14; vi. 1, 13, 14, 18, 20, cor. 2.)

Let  $ABC$  be the given rectilineal figure, to which the figure to be described is required to be similar, and  $D$  that to which it must be equal.

It is required to describe a rectilineal figure similar to  $ABC$ , and equal to  $D$ .



## CONSTRUCTION

Upon the straight line  $BC$  describe the parallelogram  $BE$  equal to the figure  $ABC$ ; (i. 45 cor.)

also upon  $CE$  describe the parallelogram  $CM$ , equal to  $D$ , and having the angle  $FCE$  equal to the angle  $CBL$ ; (i. 45 cor.)

therefore  $BC$  and  $CF$  are in a straight line, as also  $LE$  and  $EM$ , (i. 29. and i. 14.)

between BC and CF find a mean proportional GH ; (VI. 13.)  
 and upon GH describe the rectilineal figure KGH  
 similar and similarly situated to the figure ABC ; (VI. 18.)

**Then KGH shall be the rectilineal figure required.**

#### DEMONSTRATION

Because BC is to GH as GH to CF,  
 and that if three straight lines be proportionals, as the first is to the third, so is the figure upon the first to the similar and similarly described figure upon the second ; (VI. 20, cor. 2.)

*therefore, as BC to CF, so is the rectilineal figure ABC to KGH ;*

but as BC to CF, so is the parallelogram BE to the parallelogram EF ; (VI. 1.)

*therefore as the rectilineal figure ABC is to KGH, so is the parallelogram BE to the parallelogram EF ; (v. 11.)*

And the rectilineal figure ABC is equal to the parallelogram BE ; (constr.)

*therefore the rectilineal figure KGH is equal to the parallelogram EF ; (v. 14.)*

but EF is equal to the figure D ; (constr.)

*wherefore also KGH is equal to D ;*

and it is similar to ABC ;

**therefore the rectilineal figure KGH has been described similar to the figure ABC, and equal to D. Q. E. F.**

#### PROP. XXVI.—THEOREM.

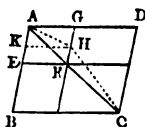
*If two similar parallelograms have a common angle, and be similarly situated ;*

*then they are about the same diameter.*

(References—Prop. v. 9, 11 ; VI. 24, def. 1.)

Let the parallelograms ABCD, AEFG be similar and similarly situated, and have the angle DAB common.

**Then ABCD and AEFG shall be about the same diameter.**



## CONSTRUCTION

For, if not, let, if possible, the parallelogram BD have its diameter AHC in a different straight line from AF, the diameter of the parallelogram EG,  
 and let GF meet AHC in H;  
 and through H draw HK parallel to AD or BC.

## DEMONSTRATION

Since the parallelograms ABCD, AKHG are assumed to be about the same diameter, they must be similar to one another; (VI. 24.)

*therefore as DA to AB, so must GA be to AK; (VI. def. 1.)*

but because ABCD and AEFB are similar parallelograms; (hyp.)

*therefore as DA is to AB, so is GA to AE;*

*therefore as GA to AE, so must GA be to AK; (v. 11.)*

that is, GA must have the same ratio to each of the straight lines AE, AK;

*and therefore AK must be equal to AE; (v. 9.)*

the less equal to the greater, which is impossible;

*therefore ABCD and AKHG are not about the same diameter;*

**wherefore ABCD and AEFB are about the same diameter.**

Therefore, if two similar, &c.

Q. E. D.

## PROP. XXVII.—THEOREM.

*Of all parallelograms applied to the same straight line, and deficient by parallelograms, similar and similarly situated to that which is described upon the half of the line; that which is applied to the half, and is similar to its defect, is the greatest.*

(References—Prop. I. 34, 36, 43; VI. 26.)

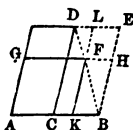
Let AB be a straight line divided into two equal parts in C; and let the parallelogram AD be applied to the half AC, which is therefore deficient from the parallelogram upon the whole line AB by the parallelogram CE upon the other half CB.

Of all the parallelograms applied to any other parts of AB, and deficient by parallelograms that are similar and similarly situated to CE, AD shall be the greatest.

Let AF be any parallelogram applied to AK, any other part of AB than the half, so as to be deficient from the parallelogram upon the

whole line AB by the parallelogram KH similar and similarly situated to CE.

**Then AD shall be greater than AF**



*First*, let AK the base of AF, be greater than AC the half of AB.

#### CONSTRUCTION

Because CE is similar to the parallelogram HK ; (hyp.)

they are about the same diameter ; (vl. 26.)

draw their diameter DB, and complete the diagram.

#### DEMONSTRATION

Then, because the parallelogram CF is equal to FE ; (l. 43.)

add KH to both ;

*therefore the whole CH is equal to the whole KE ;*

but CH is equal to CG, because the base AC is equal to the base CB ;  
(l. 36.)

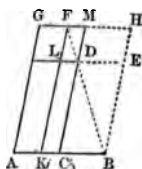
*therefore CG is equal to KE ; (ax. 1.)*

to each of these equals add CF ;

*then the whole AF is equal to the gnomon CHL ; (ax. 2.)*

**therefore CE or the parallelogram AD is greater than the parallelogram AF.**

*Secondly*, let AK the base of AF be less than AC.



#### DEMONSTRATION

Then, the same construction being made,

*because BC is equal to CA,*

*therefore HM is equal to MG ; (I. 34.)*

therefore, the parallelogram DH is equal to the parallelogram DG  
(I. 36.)

*wherefore DH is greater than LG ;*

but DH is equal to DK ; (I. 43.)

*therefore DK is greater than LG ;*

to each of these add AL ;

**therefore the whole AD is greater than the whole AF.**

Therefore, of all parallelograms, &c.

Q. E. D.

### PROP. XXVIII.—PROBLEM.

*To a given straight line to apply a parallelogram equal to a given rectilinear figure, and deficient by a parallelogram similar to a given parallelogram ; but the given rectilinear figure to which the parallelogram to be applied is to be equal, must not be greater than the parallelogram applied to half of the given line, having its defect similar to the defect of that which is to be applied ; that is, to the given parallelogram.*

(References—Prop. I. 10, 34, 36, 43 ; VI. 18, 21, 22, 25, 26, 27, 36.)

Let AB be the given straight line, and C the given rectilinear figure, to which the parallelogram to be applied is required to be equal, which figure must not be greater (VI. 27) than the parallelogram applied to half of the line, having its defect from that upon the whole line similar to the defect of that which is to be applied ; and let D be the parallelogram to which this defect is required to be similar.

It is required to apply a parallelogram to the straight line AB, which shall be equal to the figure C, and be deficient from the parallelogram upon the whole line by a parallelogram similar to D.

#### CONSTRUCTION

Divide AB into two equal parts in the point E ; (I. 10.)

and upon EB describe the parallelogram ECFG similar and similarly situated to D ; (VI. 18.)

and complete the parallelogram AG, which must either be equal to C, or greater than it, by the determination.

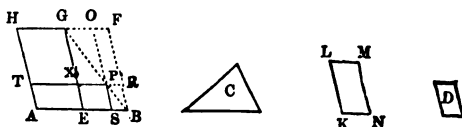
If AG be equal to C, then what was required is already done ;

for, upon the straight line AB, the parallelogram AG is applied equal, to the figure C, and deficient by the parallelogram EF similar to D.

But, if AG be not similar to C, it is greater than it ;

and  $EF$  is equal to  $AG$  ; (I. 36.)

therefore  $EF$  also is greater than  $C$  ;



make the parallelogram  $KLMN$  equal to the excess of  $EF$  above  $C$ , and similar and similarly situated to  $D$  ; (vi. 25.)

and since  $D$  is similar to  $EF$  ; (constr.)

therefore also  $KM$  is similar to  $EF$  ; (vi. 21.)

let  $KL$  be the homologous side to  $EG$ , and  $LM$  to  $GF$  ;

and because  $EF$  is equal to  $C$ , and  $KM$  together,  $EF$  is greater than  $KM$  ;

therefore the straight line  $EG$  is greater than  $KL$ , and  $GF$  than  $LM$  ;

make  $GX$  equal to  $LK$ , and  $GO$  equal to  $LM$  ; (I. 3.)

and complete the parallelogram  $XGOP$  ; (I. 31.)

therefore  $XO$  is equal and similar to  $KM$  ;

but  $KM$  is similar to  $EF$  ;

therefore also  $XO$  is similar to  $KM$ ,

wherefore  $XO$  and  $EF$  are about the same diameter ; (vi. 26.)

let  $GPB$  be their diameter, and complete the scheme.

**Then  $TS$  shall be the parallelogram required.**

#### DEMONSTRATION

Because  $EF$  is equal to  $C$  and  $KM$  together,

and  $XO$  a part of the one is equal to  $KM$  a part of the other,

therefore the remainder, viz. the gnomon  $ERO$ , is equal to the remainder  $C$  ; (ax. 3.)

and because  $OR$  is equal to  $XS$ , by adding  $SR$  to each,

the whole  $OB$  is equal to the whole  $XB$  ; (I. 43.)

but  $XB$  is equal to  $TE$ , because the base  $AE$  is equal to the base  $EB$  ; (I. 36.)

wherefore also  $TE$  is equal to  $OB$  ; (ax. 1.)

add  $XS$  to each,

then the whole  $TS$  is equal to the whole, viz. to the gnomon  $ERO$  ;

but it has been proved that the gnomon  $ERO$  is equal to  $C$  ;

therefore also **TS** is equal to **C**.

Wherefore the parallelogram **TS**, equal to the given rectilinear figure **C**, is applied to the given straight line **AB**, deficient by the parallelogram **SR**, similar to the given one **D**, because **SR** is similar to **EF**; (vi. 24.) Q. E. F.

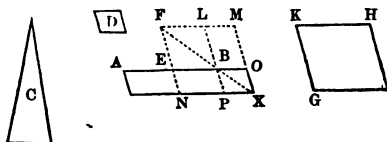
### PROP. XXIX.—PROBLEM.

*To a given straight line to apply a parallelogram equal to a given rectilinear figure, exceeding by a parallelogram similar to another given.*

(References—Prop. I. 10, 34, 36, 43 ; VI. 18, 21, 24, 25, 26.)

Let **AB** be the given straight line, and **C** the given rectilinear figure to which the parallelogram to be applied is required to be equal, and **D** the parallelogram to which the excess of the one to be applied above that upon the given line is required to be similar.

It is required to apply a parallelogram to the given straight line **AB** which shall be equal to the figure **C**, exceeding by a parallelogram similar to **D**.



### CONSTRUCTION

Divide **AB** into two equal parts in the point **E**; (I. 10.)

and upon **EB** describe the parallelogram **EL** similar and similarly situated to **D**; (VI. 18.)

and make the parallelogram **GH** equal to **EL** and **C** together, and similar and similarly situated to **D**; (VI. 25.)

therefore **GH** is similar to **EL**; (VI. 21.)

let **KH** be the side homologous to **FL**, and **KG** to **FE**.

And because the parallelogram **GH** is greater than **EL**,

therefore the side **KH** is greater than **FL**, and **KG** than **FE**;

produce **FL** and **FE**, and make **FLM** equal to **KH**, and **FEN** to **KG**, and complete the parallelogram **MN**;

therefore **MN** is equal and similar to **GH**;

but **GH** is similar to **EL**;



therefore  $MN$  is similar to  $EL$  ;

wherefore  $EL$  and  $MN$  are about the same diameter ; (VI. 26.)

draw their diameter  $FX$  and complete the scheme.

**Then  $AX$  shall be the parallelogram required.**

#### DEMONSTRATION

Since  $GH$  is equal to  $EL$  and  $C$  together, and that  $GH$  is equal to  $MN$  ;

therefore  $MN$  is equal to  $EL$  and  $C$  ;

take away the common part  $EL$  ;

then the remainder, viz. the gnomon  $NOL$ , is equal to  $C$ .

And because  $AE$  is equal to  $EB$ ,

the parallelogram  $AN$  is equal to the parallelogram  $NB$ , (I. 36.) that is, to  $BM$  ; (I. 43.)

add  $NO$  to each ;

therefore the whole, viz. the parallelogram  $AX$ , is equal to the gnomon  $NOL$  ;

but the gnomon  $NOL$  is equal to  $C$  ;

therefore also  $AX$  is equal to  $C$ .

Wherefore to the straight line  $AB$  there is applied the parallelogram  $AX$  equal to the given rectilineal figure  $C$ , exceeding by the parallelogram  $PO$ , which is similar to  $D$ , because  $PO$  is similar to  $EL$ .  
(VI. 24.) Q. E. F.

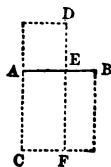
#### PROP. XXX.—PROBLEM.

*To cut a given straight line in extreme and mean ratio.*

(References—Prop. I. 34, 46 ; II. 11 ; V. 14 ; VI. 14, 17, 29, def. 3.)

Let  $AB$  be the given straight line.

It is required to cut it in extreme and mean ratio.



## CONSTRUCTION

Upon  $AB$  describe the square  $BC$ ; (I. 46.)  
and to  $AC$  apply the parallelogram  $CD$ , equal to  $BC$ , exceeding by the figure  $AD$  similar to  $BC$ ; (VI. 29.)

**Then  $AB$  shall be cut in extreme and mean ratio in the point  $E$ .**

## DEMONSTRATION

Because  $BC$  is a square,

*therefore also  $AD$  is a square;*

and because  $BC$  is equal to  $CD$ ,

by taking the common part  $CE$  from each,

*therefore the remainder  $BF$  is equal to the remainder  $AD$ ;*

and these figures are equiangular,

*therefore their sides about the equal angles are reciprocally proportional;* (VI. 14.)

*therefore, as  $FE$  to  $ED$ , so is  $AE$  to  $EB$ .*

but  $FE$  is equal to  $AC$  (I. 34.), that is, to  $AB$  (I. def. 30.), and  $ED$  is equal to  $AE$ ;

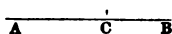
*therefore as  $BA$  to  $AE$ , so is  $AE$  to  $EB$ ;*

but  $AB$  is greater than  $AE$ ;

*wherefore  $AE$  is greater than  $EB$ ;* (v. 14.)

**therefore the straight line  $AB$  is cut in extreme and mean ratio in  $E$ .** (VI. def. 3.)

Q. E. F.



Otherwise.

## CONSTRUCTION

Divide  $AB$  in the point  $C$ , so that the rectangle contained by  $AB$ ,  $BC$ , may be equal to the square of  $AC$ ; (II. 11.)

**Then  $AB$  shall be cut in extreme and mean ratio in  $C$ .**

## DEMONSTRATION

Because the rectangle  $AB$ ,  $BC$  is equal to the square of  $AC$ ;

*therefore, as  $BA$  to  $AC$ , so is  $AC$  to  $CB$ ;* (VI. 17.)

**therefore  $AB$  is cut in extreme and mean ratio in  $C$ .**  
(VI. def. 3.)

Q. E. F.

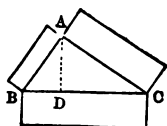
## PROP. XXXI.—THEOREM.

*In right-angled triangles, the rectilinear figure described upon the side opposite to the right angle, is equal to the similar and similarly described figures upon the sides containing the right angle.*

(References—Prop. I. 12, 24; v. A, B, 24; vi. 4, 8, 20, cor. 2.)

Let ABC be a right-angled triangle, having the right angle BAC.

**Then the rectilinear figure described upon BC shall be equal to the similar and similarly described figures upon BA, AC.**



CONSTRUCTION

Draw the perpendicular AD; (I. 12.)

## DEMONSTRATION

Because in the right-angled triangle ABC, AD is drawn from the right angle at A perpendicular to the base BC,

*the triangles ABD, ADC are similar to the whole triangle ABC, and to one another ; (vi. 8.)*

and because the triangle ABC is similar to ADB,

*therefore as CB to BA, so is BA to BD; (vi. 4.)*

and because these three straight lines are proportionals, as the first is to the third, so is the figure upon the first to the similar and similarly described figure upon the second ; (vi. 20, cor. 2.)

*therefore as CB to BD, so is the figure upon CB to the similar and similarly described figure upon BA ;*

and inversely,

*as DB to BC, so is the figure upon BA to that upon BC. (v. B.)*

For the same reason, as DC to CB, so is the figure upon CA to that upon CB ;

*therefore as BD and DC together are to BC, so are the figures upon BA, AC to that upon BC ; (v. 24.)*

but BD and DC together are equal to BC;

therefore the figure described on  $BC$  is equal to the similar and similarly described figures on  $BA$ ,  $AC$ .  
(v. A.)

Wherefore, in right-angled triangles, &c

Q. E. D.

PROP. XXXII.—THEOREM.

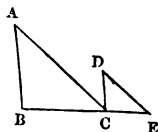
*If two triangles which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another ;*

*then the remaining sides shall be in a straight line.*

(References—Prop. I. 14, 29, 32; VI. 6.)

Let  $ABC$ ,  $DCE$  be two triangles which have the two sides  $BA$ ,  $AC$  proportional to the two  $CD$ ,  $DE$ , viz.  $BA$  to  $AC$ , as  $CD$  to  $DE$  ; and let  $AB$  be parallel to  $DC$ , and  $AC$  to  $DE$ .

**Then  $BC$  and  $CE$  shall be in a straight line.**



DEMONSTRATION

Because  $AB$  is parallel to  $DC$ , and the straight line  $AC$  meets them,  
*the alternate angles  $BAC$ ,  $ACD$  are equal ;* (I. 29.)

for the same reason, the angle  $CDE$  is equal to the angle  $ACD$  ;

*wherefore also  $BAC$  is equal to  $CDE$ .* (ax. 1.)

And because the triangles  $ABC$ ,  $DCE$  have one angle at  $A$  equal to one at  $D$ ,

and the sides about these angles proportionals, viz.  $BA$  to  $AC$ , as  $CD$  to  $DE$  ;

therefore the triangle  $ABC$  is equiangular to the triangle  $DCE$  ;  
(vi. 6.)

*therefore the angle  $ABC$  is equal to the angle  $DCE$  ;*

and the angle  $BAC$  was proved to be equal to  $ACD$  ;

*therefore the whole angle  $ACE$  is equal to the two angles  $ABC$ ,  $BAC$  ;*  
(ax. 2.)

add to each of these equals the common angle  $ACB$ ,

then the angles  $\angle ACE$ ,  $\angle ACB$  are equal to the angles  $\angle ABC$ ,  $\angle BAC$ ,  $\angle ACB$  ;  
but  $\angle ABC$ ,  $\angle BAC$ ,  $\angle ACB$  are equal to two right angles ; (I. 32.)

therefore also the angles  $\angle ACE$ ,  $\angle ACB$  are equal to two right angles.

And since at the point  $C$ , in the straight line  $AC$ , the two straight lines  $BC$ ,  $CE$ , which are on the opposite sides of it, make the adjacent angles  $\angle ACE$ ,  $\angle ACB$  equal to two right angles ;

therefore  $BC$  and  $CE$  are in a straight line. (I. 14.)

Wherefore, if two triangles, &c.

Q. E. D.

### PROP. XXXIII.—THEOREM.

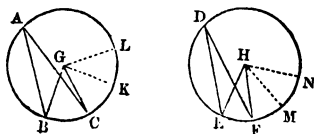
*In equal circles, angles, whether at the centres or circumferences, have the same ratio which the circumferences on which they stand have to one another ; so also have the sectors.*

(References—Prop. I. 4 ; III. 20, 24, 27, def. 11 ; V. 15, def. 5.)

Let  $ABC$ ,  $DEF$  be equal circles ; and let  $BGC$  and  $EHF$  be angles at their centres, and  $BAC$  and  $EDF$  angles at their circumferences.

**Then as the circumference  $BC$  to the circumference  $EF$ ,  
so shall the angle  $BGC$  be to the angle  $EHF$ , and the  
angle  $BAC$  to the angle  $EDF$  ;**

**and also the sector  $BGC$  to the sector  $EHF$ .**



### CONSTRUCTION

Take any number of circumferences  $CK$ ,  $KL$ , each equal to  $BC$ , and  
any number whatever  $FM$ ,  $MN$ , each equal to  $EF$  ;  
and join  $GK$ ,  $GL$ ,  $HM$ ,  $HN$ .

### DEMONSTRATION

Because the circumferences  $BC$ ,  $CK$ ,  $KL$  are all equal,

the angles  $BGC$ ,  $CGK$ ,  $KGL$  are also all equal ; (III. 27.)

therefore what multiple soever the circumference  $BL$  is of the circumference  $BC$ , the same multiple is the angle  $BGL$  of the angle  $BGC$  ;

for the same reason, whatever multiple the circumference  $EN$  is of the circumference  $EF$ , the same multiple is the angle  $EHN$  of the angle  $EHF$ .

And if the circumference BL be equal to the circumference EN, the angle BGL is also equal to the angle EHN ; (III. 27.)

and if the circumference BL be greater than EN, likewise the angle BGL is greater than EHN ; and if less, less.

Since, then, there are four magnitudes, the two circumferences BC, EF, and the two angles BGC, EHF ;

and that of the circumference BC, and of the angle BGC, have been taken any equimultiples whatever, viz. the circumference BL, and the angle BGL ;

and of the circumference EF, and of the angle EHF have been taken any equimultiples whatever, viz. the circumference EN, and the angle EHN ;

and since it has been proved, that if the circumference BL be greater than EN, the angle BGL is greater than EHN ; and if equal, equal ; and if less, less ;

*therefore as the circumference BC to the circumference EF, so is the angle BGC to the angle EHF ; (v. def. 5.)*

but as the angle BGC is to the angle EHF, so is the angle BAC to the angle EDF ; (v. 15.) for each is double of each ; (III. 20.)

**therefore, as the circumference BC is to EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF.**

Also, as the circumference BC to EF, so shall the sector BGC be to the sector EHF.

#### CONSTRUCTION\*

Join BC, CK, and in the circumferences, BC, CK, take any points X, O, and join BX, XC, CO, OK.

#### DEMONSTRATION

Then, in the triangles GBC, GCK,

because the two sides BG, GC, are equal to the two CG, GK each to each,

and that they contain equal angles ;

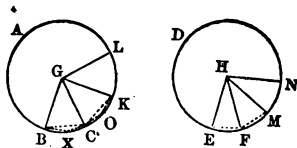
*therefore the base BC is equal to the base CK, and the triangle GBC to the triangle GCK ; (I. 4.)*

and because the circumference BC is equal to the circumference CK,

*the remaining part of the whole circumference of the circle ABC, is equal to the remaining part of the whole circumference of the same circle ; (ax. 3.)*

\* See figs, next page.

therefore the angle  $BXC$  is equal to the angle  $COK$  ; (III. 27.)  
 therefore the segment  $BXC$  is similar to the segment  $COK$  ; (III. def. 11.)  
 and they are upon equal straight lines,  $BC$ ,  $CK$  ;



but similar segments of circles upon equal straight lines, are equal to one another ; (III. 24.)

therefore the segment  $BXC$  is equal to the segment  $COK$  ;  
 and the triangle  $BGC$  was proved to be equal to the triangle  $CGK$  ;  
 therefore the whole, the sector  $BGC$ , is equal to the whole, the sector  $CGK$  ;  
 for the same reason, the sector  $KGL$  is equal to each of the sectors  $BGC$ ,  $CGK$ .

In the same manner, the sectors  $EHF$ ,  $FHM$ ,  $MHN$  may be proved equal to one another ;

therefore, what multiple soever the circumference  $BL$  is of the circumference  $BC$ , the same multiple is the sector  $BGL$  of the sector  $BGC$  ;  
 and for the same reason, whatever multiple the circumference  $EN$  is of  $EF$ , the same multiple is the sector  $EHN$  of the sector  $EHF$  ;  
 and if the circumference  $BL$  be equal to  $EN$ , the sector  $BGL$  is equal to the sector  $EHN$  ;

and if the circumference  $BL$  be greater than  $EN$ , the sector  $BGL$  is greater than the sector  $EHN$ , and if less, less.

Since, then, there are four magnitudes, the two circumferences  $BC$ ,  $EF$ , and the two sectors  $BGC$ ,  $EHF$ ,

and of the circumference  $BC$ , and sector  $BGC$ , the circumference  $BL$  and sector  $BGL$  are any equimultiples whatever ;

and of the circumference  $EF$ , and sector  $EHF$ , the circumference  $EN$ , and sector  $EHN$  are any equimultiples whatever ;

and since it has been proved, that if the circumference  $BL$  be greater than  $EN$ , the sector  $BGL$  is greater than the sector  $EHN$  ; and if equal, equal ; and if less, less ;

therefore, as the circumference  $BC$  is to the circumference  $EF$ , so is the sector  $BGC$  to the sector  $EHF$ .  
 (v. def. 5.)

Wherefore, in equal circles, &c.

Q. E. D.

## PROP. B.—THEOREM.

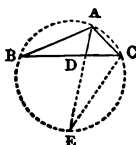
*If an angle of a triangle be bisected by a straight line which likewise cuts the base;*

*then the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line which bisects the angle.*

(References—Prop. I. 32 ; II. 3 ; III. 21, 35 ; IV. 5 ; VI. 4, 16.)

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD.

**Then the rectangle BA, AC shall be equal to the rectangle BD, DC, together with the square of AD.**



## CONSTRUCTION

Describe the circle ACB about the triangle ; (iv. 5.)  
and produce AD to the circumference in E, and join EC.

## DEMONSTRATION

Then because the angle BAD is equal to the angle CAE; (hyp.)  
and the angle ABD to the angle AEC, for they are in the same segment ; (III. 21.)

therefore the triangles ABD, AEC are equiangular to one another ;  
(I. 32.)

therefore as BA to AD, so is EA to AC ; (vi. 4.)

and consequently the rectangle BA, AC is equal to the rectangle EA, AD ; (vi. 16.)

that is, to the rectangle ED, DA, together with the square of AD ;  
(II. 3.)

but the rectangle ED, DA is equal to the rectangle BD, DC ; (III. 35.)

**therefore the rectangle BA, AC is equal to the rectangle BD, DC, together with the square of AD.**

Wherefore, if an angle, &c.

Q. E. D.



## PROP. C.—THEOREM.

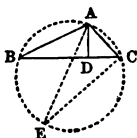
*If from any angle of a triangle, a straight line be drawn perpendicular to the base ;*

*then the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.*

(References—Prop. III. 21, 31 ; IV. 5 ; VI. 4, 16.)

Let  $ABC$  be a triangle, and  $AD$  the perpendicular from the angle  $A$  to the base  $BC$ .

**Then the rectangle  $BA, AC$  shall be equal to the rectangle contained by  $AD$  and the diameter of the circle described about the triangle.**



## CONSTRUCTION

Describe the circle  $ACB$  about the triangle ; (IV. 5.)  
draw its diameter  $AE$ , and join  $EC$ .

## DEMONSTRATION

Because the right angle  $BDA$  is equal to the angle  $BCA$  in a semi-circle ; (III. 31.)

and the angle  $ABD$  equal to the angle  $AEC$  in the same segment ; (III. 21.)

therefore the triangles  $ABD, AEC$  are equiangular ;

therefore as  $BA$  to  $AD$ , so is  $EA$  to  $AC$  ; (VI. 4.)

**and consequently the rectangle  $BA, AC$  is equal to the rectangle  $BA, AD$ .** (VI. 16.)

If, therefore, from any angle, &c.

Q. E. D.

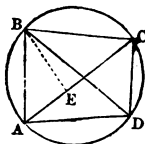
## PROP. D.—THEOREM.

*The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle, is equal to both the rectangles contained by its opposite sides.*

(References—Prop. I. 23 ; II. 1 ; III. 21 ; VI. 4, 16.)

Let  $ABCD$  be any quadrilateral figure inscribed in a circle, and join  $AC$ ,  $BD$ .

**Then the rectangle contained by  $AC$ ,  $BD$  shall be equal to the two rectangles contained by  $AB$ ,  $CD$ , and by  $AD$ ,  $BC$ .**



#### CONSTRUCTION

**Make the angle  $ABE$  equal to the angle  $DBC$  ; (I. 23.)**

#### DEMONSTRATION

Because the angle  $ABE$  is equal to the angle  $DBC$ ,  
add to each of these equals the common angle  $EBD$  ;

*then the angle  $ABD$  is equal to the angle  $EBC$  ;*

and the angle  $BDA$  is equal to the angle  $BCE$ , because they are in the same segment ; (III. 21.)

*therefore the triangle  $ABD$  is equiangular to the triangle  $BCE$  ;*

*wherefore, as  $BC$  is to  $CE$ , so is  $BD$  to  $DA$  ; (VI. 4.)*

*and consequently the rectangle  $BC$ ,  $AD$  is equal to the rectangle  $BD$ ,  $CE$ . (VI. 16.)*

Again, because the angle  $ABE$  is equal to the angle  $DBC$ , and the angle  $BAE$  to the angle  $BDC$  ; (III. 21.)

*therefore the triangle  $ABE$  is equiangular to the triangle  $BCD$  ;*

*therefore as  $BA$  to  $AE$ , so is  $BD$  to  $DC$  ;*

*wherefore the rectangle  $BA$ ,  $DC$  is equal to the rectangle  $BD$ ,  $AE$  ;*

but the rectangle  $BC$ ,  $AD$  has been shown to be equal to the rectangle  $BD$ ,  $CE$  ;

**therefore the whole rectangle  $AC$ ,  $BD$  is equal to the rectangle  $AB$ ,  $DC$ , together with the rectangle  $AD$ ,  $BC$ . (II. 1.)**

Therefore the rectangle, &c.

Q. E. D.

## BOOK XI.

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### DEFINITIONS.

#### I.

A SOLID is that which hath length, breadth, and thickness.

#### II.

That which bounds a solid is a superficies.

#### III.

A straight line is perpendicular, or at right angles to a plane, when it makes right angles with every straight line meeting it in that plane.

#### IV.

A plane is perpendicular to a plane, when the straight lines drawn in one of the planes perpendicular to the common section of the two planes are perpendicular to the other plane.

#### V.

The inclination of a straight line to a plane, is the acute angle contained by that straight line, and another drawn from the point in which the first line meets the plane, to the point in which a perpendicular to the plane drawn from any point of the first line above the plane, meets the same plane.

#### VI.

The inclination of a plane to a plane, is the acute angle contained by two straight lines drawn from any the same point of their common section at right angles to it, one upon one plane, and the other upon the other plane.

#### VII.

Two planes are said to have the same, or a like inclination to one another, which two other planes have, when the said angles of inclination are equal to one another.

## VIII.

Parallel planes are such as do not meet one another though produced.

## IX.

A solid angle is that which is made by the meeting, in one point, of more than two plane angles, which are not in the same plane.

## X.

Equal and similar solid figures are such as are contained by similar planes equal in number and magnitude.

## XI.

Similar solid figures are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes.

## XII.

A pyramid is a solid figure contained by planes that are constituted betwixt one plane and one point above it in which they meet.

## XIII.

A prism is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another; and the others parallelograms.

## XIV.

A sphere is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved.

## XV.

The axis of a sphere is the fixed straight line about which the semicircle revolves.

## XVI.

The centre of a sphere is the same with that of the semicircle.

## XVII.

The diameter of a sphere is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

## XVIII.

A cone is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone; if it be less than the other side, an obtuse-angled; and if greater, an acute-angled cone.

**XIX.**

The axis of a cone is the fixed straight line about which the triangle revolves.

**XX.**

The base of a cone is the circle described by that side containing the right angle, which revolves.

**XXI.**

A cylinder is a solid figure described by the revolution of a right-angled parallelogram about one of its sides which remains fixed.

**XXII.**

The axis of a cylinder is the fixed straight line about which the parallelogram revolves.

**XXIII.**

The bases of a cylinder are the circles described by the two revolving opposite sides of the parallelogram.

**XXIV.**

Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.

**XXV.**

A cube is a solid figure contained by six equal squares.

**XXVI.**

A tetrahedron is a solid figure contained by four equal and equilateral triangles.

**XXVII.**

An octahedron is a solid figure contained by eight equal and equilateral triangles.

**XXVIII.**

A duodecahedron is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.

**XXIX.**

An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

**DEF. A.**

A parallelopiped is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.

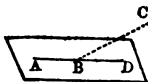
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## PROP. I.—THEOREM.

*One part of a straight line cannot be in a plane, and another part above it.*

(References—Prop. I. 11 cor., def. 7.)

If it be possible, let AB, part of the straight line ABC, be in the plane, and the part BC above it.



## CONSTRUCTION

Since the straight line AB is in the plane, it can be produced in that plane;

let it be produced to D;

and let any plane pass through the straight line AD, and be turned about it, until it pass through the point C.

## DEMONSTRATION

Then, because the points B, C are assumed to be in the same plane, the straight line BC must be in it; (i. def. 7.)

*therefore there can be two straight lines ABC, ABD in the same plane that have a common segment AB; (i. 11 cor.)*

which is impossible.

Therefore one part, &c.

Q. E. D.

## PROP. II.—THEOREM.

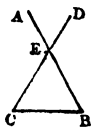
*Two straight lines which cut one another are in one plane, and three straight lines which meet one another are in one plane.*

(References—Prop. XI. 1.)

Let two straight lines AB, CD cut one another in E.

**Then AB, CD shall be in one plane;**

**and three straight lines EC, EB, AB, which meet one another shall be in one plane.**



## CONSTRUCTION

Let any plane pass through the straight line EB, and let the plane be turned about EB, produced if necessary, until it pass through the point C.

## DEMONSTRATION

Then, because the points E, C, are in this plane,  
*therefore the straight line EC is in it ; (I. def. 7.)*

for the same reason, the straight line BC is in the same;  
 and by the hypothesis, EB is in it ;

**therefore the three straight lines EC, CB, BE are in one plane.**

But in the plane in which EC, EB are, in the same are CD, AB; (XI. 1.)

**therefore, AB, CD are in one plane.**

Wherefore two straight lines, &c.

Q. E. D.

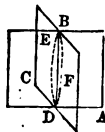
## PROP. III.—THEOREM.

*If two planes cut one another, their common section is a straight line.*

(Reference—I. ax. 10.)

Let two planes AB, BC cut one another, and let the line DB be their common section.

**Then DB shall be a straight line.**



## CONSTRUCTION

If it be not, from the point D to B, draw in the plane AB, the straight line DEB, and in the plane BC, the straight line DFB. (post. 1.)

## DEMONSTRATION

Because the two lines DEB, DFB, which are assumed to be straight lines, have the same extremities ; (constr.)

*therefore they must include a space betwixt them ;*  
 which is impossible; (I. ax. 10.)

**therefore ED the common section of the planes AB, BC,**  
**cannot but be a straight line.**

Wherefore, if two planes, &c.

Q. E. D.

### PROP. IV.—THEOREM.

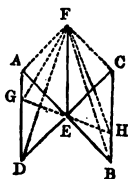
*If a straight line stand at right angles to each of two straight lines in the point of their intersection ;*

*then it shall also be at right angles to the plane which passes through them, that is, to the plane in which they are.*

(References—Prop. I. 4, 8, 15, 26; XI. def. 3.)

Let the straight line EF stand at right angles to each of the straight lines AB, CD, in E the point of their intersection.

**Then EF shall also be at right angles to the plane passing through AB, CD.**



### CONSTRUCTION

Take the straight lines AE, EB, CE, ED all equal to one another;  
 and through E draw, in the plane in which are AB, CD, any straight  
 line GEH, and join AD, CB ;

and from any point F, in EF, draw FA, FG, FD, FC, FH, FB.

### DEMONSTRATION

Because the two straight lines AE, ED are equal to the two BE, EC,  
 each to each ;

and that they contain equal angles AED, BEC ; (I. 15.)

*therefore the base AD is equal to the base BC, and the angle DAE to the angle EBC ; (I. 4.)*

and the angle AEG is equal to the angle BEH. (I. 15.)

Hence in the two triangles AEG, BEH, there are two angles of the one  
 equal to two angles of the other, each to each,



and the sides AE, EB, adjacent to the equal angles, equal to one another ;

*therefore then other sides are equal, that is GE is equal to EH, and AG to BH ; (I. 26.)*

and because AE is equal to EB, and FE common and at right angles to them,

*therefore the base AF is equal to the base FB ; (I. 4.)*

for the same reason, CF is equal to FD.

And because AD is equal to BC, and AF to FB,

the two sides FA, AD are equal to the two FB, BC, each to each ;

and the base DF was proved equal to the base FC ;

*therefore the angle FAD is equal to the angle FBC. (I. 8.)*

Again, it was proved that GA is equal to BH, and also AF to FB ;

*therefore FA and AG are equal to FB and BH, each to each ;*

and the angle FAG has been proved equal to the angle FBH ;

*therefore the base GF is equal to the base FH. (I. 4.)*

Again, because it was proved that GE is equal to EH, and EF is common ;

GE, EF are equal to HE, EF ;

and the base GF is equal to the base FH ;

*therefore the angle GEF is equal to the angle HEF ; (I. 8.)*

and consequently each of these angles is a right angle ; (I. def. 10.)

*therefore FE makes right angles with GH, that is, with any straight line drawn through E in the plane passing through AB, CD.*

In like manner, it may be proved, that FE makes right angles with every straight line which meets it in that plane ;

but a straight line is at right angles to a plane, when it makes right angles with every straight line which meets it in that plane ; (XI. def. 3.)

**therefore EF is at right angles to the plane in which are AB, CD.**

Wherefore, if a straight line, &c.

Q. E. D.

### PROP. V.—THEOREM.

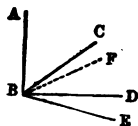
*If three straight lines meet all in one point, and a straight line stands at right angles to each of them in that point ;*

*then these straight lines are in one and the same plane.*

(References—Prop. XI. 3, 4, def. 3.)

Let the straight line AB stand at right angles to each of the straight lines BC, BD, BE, in B the point where they meet.

**Then BC, BD, BE, shall be in one and the same plane.**



## CONSTRUCTION

If not, let, if possible, BD and BE be in one plane, and BC be above it; and let a plane pass through AB, BC, the common section of which with the plane, in which BD and BE are, is a straight line; (XI. 3.)

let this straight line be BF;

then the three straight lines AB, BC, BF must be all in one plane, viz. that which passes through AB, BC.

## DEMONSTRATION

Because AB stands at right angles to each of the straight lines BD, BE,

therefore it is also at right angles to the plane passing through them; (XI. 4.)

*wherefore it makes right angles with every straight line meeting it in that plane; (XI. def. 3.)*

but BF which is assumed to be in that plane meets it;

*therefore the angle ABF must be a right angle;*

but the angle ABC is also a right angle; (hyp.)

*therefore the angle ABF must be equal to the angle ABC, and they must be in the same plane;*

which is impossible; (I. ax. 9.)

*therefore the straight line BC is not above the plane in which are BD and BE;*

**wherefore the three straight lines BC, BD, BE are in one and the same plane.**

Wherefore, if three straight lines, &c.

Q. E. D.

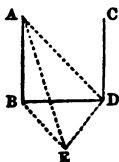
## PROP. VI.—THEOREM.

*If two straight lines be at right angles to the same plane, they shall be parallel to one another.*

(References—Prop. I. 3, 4, 8, 11, 28 ; XI. 2, 5, def. 3.)

Let the straight lines AB, CD be at right angles to the same plane.

**Then AB shall be parallel to CD.**



## CONSTRUCTION

Let them meet in the plane in the points B, D ; join BD ;  
draw DE at right angles, in the same plane ; (I. 11.)  
make DE equal to AB ; (I. 3.) and join BE, AE, AD.

## DEMONSTRATION

Because AB is perpendicular to the plane,  
*therefore it makes right angles with every straight line which meets it,  
and is in that plane ; (XI. def. 3.)*

but BD, BE, which are in that plane, do each of them meet AB ;  
*therefore each of the angles ABD, ABE is a right angle ;*  
for the same reason, each of the angles CDB, CDE is a right angle.  
And in the triangles ABD, EDB,

because AB is equal to DE, and BD common,  
the two sides AB, BD are equal to the two ED, DB ;  
and they contain right angles ;

*therefore the base AD is equal to the base BE. (I. 4.)*

Again, in the triangles ABE, EDA,  
because AB is equal to DE, and BE to AD ;  
AB, BE are equal to ED, DA, each to each,  
and the base AE is common,

*therefore the angle ABE is equal to the angle EDA ; (I. 8.)*  
but ABE is a right angle ;  
*therefore EDA is also a right angle, and ED perpendicular to DA ;*

but ED is also perpendicular to each of the two BD, DC ;

therefore ED is at right angles to each of the three straight lines BD, DA, DC in the point in which they meet ;

*therefore these three straight lines are all in the same plane ; (xi. 5.)*

but AB is in the plane in which are BD, DA, because any three straight lines which meet one another are in one plane ; (xi. 2.)

*therefore AB, BD, DC are in one plane,*

and each of the angles ABD, BDC is a right angle ; (hyp.)

**therefore AB is parallel to CD. (I. 28.)**

Wherefore, if two straight lines, &c.

Q. E. D.

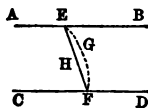
### PROP. VII.—THEOREM.

*If two straight lines be parallel, the straight line drawn from any point in the one to any point in the other, is in the same plane with the parallels.*

(References—I. ax. 10.)

Let AB, CD be parallel straight lines, and take any point E in the one, and the point F in the other.

**Then the straight line which joins E and F shall be in the same plane with the parallels.**



### CONSTRUCTION

If not, let it be, if possible, above the plane, as EGF ;

and in the plane ABCD in which the parallels are, draw the straight line EHF from E to F.

### DEMONSTRATION

Then, since EGF is assumed to be a straight line,

*the two straight lines EHF, EGF must include a space between them, which is impossible ; (I. ax. 10.)*

*therefore the straight line joining the points E, F is not above the plane in which the parallels AB, CD are,*

**therefore it is in that plane.**

Wherefore, if two straight lines, &c.

Q. E. D.

## PROP. VIII.—THEOREM.

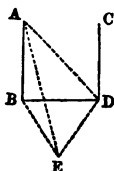
*If two straight lines be parallel, and one of them be at right angles to a plane ;*

*then the other also shall be at right angles to the same plane.*

(References—Prop. I. 3, 4, 8, 11, 29 ; XI. 4, 7, def. 3.)

Let AB, CD be two parallel straight lines, and let one of them AB be at right angles to a plane.

**Then the other CD shall be at right angles to the same plane.**



## CONSTRUCTION

Let AB, CD meet the plane in the points B, D, and join BD ;

*therefore AB, CD, BD are in one plane ; (XI. 7.)*

in the plane to which AB is at right angles, draw DE at right angles to BD. (I. 11.)

and make DE equal to AB, and join BE, AE, AD.

## DEMONSTRATION

Because AB is perpendicular to the plane, it is perpendicular to every straight line which meets it, and is in that plane ; (XI. def. 3.)

*therefore each of the angles ABD, ABE is a right angle ;*

and because the straight line BD meets the parallel straight lines AB, CD,

*therefore the angles ABD, CDB, are together equal to two right angles ; (I. 29.)*

and ABD is a right angle ; (hyp.)

*therefore also CDB is a right angle, and CD perpendicular to BD.*

And in the two triangles ABD, EDB,

because AB is equal to DE, and BD common,

the two AB, BD are equal to the two ED, DB,

and the angle ABD is equal to the angle EDB, because each of them is a right angle ;

*therefore the base AD is equal to the base BE.* (I. 4.)

Again, in the two triangles ABE, EDA,  
because AB is equal to DE, and BE to AD,  
the two AB, BE are equal to the two ED, DA,  
and the base AE is common to both triangles ;

*therefore the angle ABE is equal to the angle EDA ;* (I. 8.)

but ABE is a right angle ;

*therefore EDA is a right angle, and ED perpendicular to DA ;*

but it is also perpendicular to BD ; (constr.)

therefore ED is perpendicular to the plane which passes through BD,  
DA ; (XI. 4.)

*and therefore makes right angles with every straight line meeting it in  
that plane ;* (XI. def. 3.)

but DC is in the plane passing through BD, DA, because all three are  
in the plane in which are the parallels AB, CD,

therefore ED is at right angles to DC ;

*wherefore CD is at right angles to DE ;*

but CD is also at right angles to DB ;

*therefore CD is at right angles to the two straight lines DE, DB in the  
point of their intersection D ;*

**therefore it is at right angles to the plane passing through  
DE, DB, which is the same plane to which AE is at  
right angles.** (XI. 4.)

Therefore, if two straight lines, &c.

Q. E. D

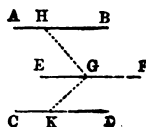
# PROP. IX.—THEOREM.

*Two straight lines which are each of them parallel to the same straight  
line, and not in the same plane with it, are parallel to one another.*

(References—Prop. I. 11 ; XI. 4, 6, 8.)

Let AB, CD be each of them parallel to EF and not in the same  
plane with it.

**Then AB shall be parallel to CD.**



## CONSTRUCTION

In  $EF$  take any point  $G$ , from which draw, in the plane passing through  $EF$ ,  $AB$ , the straight line  $GH$  at right angles to  $EF$ ; (I. 11.) and in the plane passing through  $EF$ ,  $CD$ , draw  $GK$  at right angles to the same  $EF$ .

## DEMONSTRATION

And because  $EF$  is perpendicular both to  $GH$  and  $GK$ ,  
*therefore  $EF$  is perpendicular to the plane  $HGK$  passing through them ;*  
 (XI. 4.)

and  $EF$  is parallel to  $AB$ ; (hyp.)

*therefore  $AB$  is at right angles to the plane  $HGK$ ; (XI. 8.)*

for the same reason,  $CD$  is likewise at right angles to the plane  $HGK$ ;

*therefore  $AB$ ,  $CD$  are each of them at right angles to the plane  $HGK$ ;*

but if two straight lines are at right angles to the same plane, they are parallel to one another; (XI. 6.)

**therefore  $AB$  is parallel to  $CD$ .**

Wherefore, two straight lines, &c.

Q. E. D.

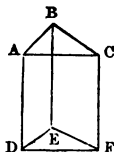
## PROP. X.—THEOREM.

*If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the first two; then the first two and the other two shall contain equal angles.*

(References—Prop. I. 8, 33; XI. 9.)

Let the two straight lines  $AB$ ,  $BC$ , which meet one another, be parallel to the two straight lines  $DE$ ,  $EF$  that meet one another, and are not in the same plane with  $AB$ ,  $BC$ .

**Then the angle  $ABC$  shall be equal to the angle  $DEF$ .**



## CONSTRUCTION

Take  $BA$ ,  $BC$ ,  $ED$ ,  $EF$  all equal to one another, and join  $AD$ ,  $CF$ ,  $BE$ ,  $AC$ ,  $DF$ .

## DEMONSTRATION

Because BA is equal and parallel to ED,

*therefore AD is both equal and parallel to BE; (I. 33.)*

for the same reason, CF is equal and parallel to BE;

*therefore AD and CF are each of them equal and parallel to BE;*

but straight lines that are parallel to the same straight line, and not in the same plane with it, are parallel to one another; (XI. 9.)

*therefore AD is parallel to CF;*

and AD is also equal to CF; (I. ax. 1.)

and AC, DF join them towards the same parts;

*therefore AC is equal and parallel to DF; (I. 33.)*

and because AB, BC are equal to DE, EF, and the base AC to the base DF;

**therefore the angle ABC is equal to the angle DEF.**  
(I. 8.)

Therefore, if two straight lines, &c.

Q. E. D.

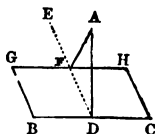
## PROP. XI.—PROBLEM.

*To draw a straight line perpendicular to a plane, from a given point above it.*

(References—Prop. I. 11, 12, 31; XI. 4, 8, def. 3.)

Let A be the given point above the plane BH.

It is required to draw from the point A a straight line perpendicular to the plane BH.



## CONSTRUCTION

In the plane draw any straight line BC, and from the point A draw AD perpendicular to BC; (I. 12.)

if then AD be also perpendicular to the plane BH, the thing required is already done;

but if it be not, from the point D draw, in the plane BH, the straight line DE, at right angles to BC; (I. 11.)



and from the point A, draw AF perpendicular to DE.

**Then AF shall be perpendicular to the plane BH.**

Through F draw GH parallel to BC. (I. 31.)

#### DEMONSTRATION

Then, because BC is at right angles to ED and DA,

*therefore BC is at right angles to the plane passing through ED, DA;*

(XI. 4.)

and GH is parallel to BC; (constr.)

but, if two straight lines be parallel, one of which is at right angles to a plane, the other is at right angles to the same plane; (XI. 8.)

*therefore GH is at right angles to the plane through ED, DA, and is perpendicular to every straight line meeting it in that plane;*

(XI. def. 3.)

but AF, which is in the plane through ED, AD, meets it,

*therefore GH is perpendicular to AF;*

*wherefore AF is perpendicular to GH;*

and AF is perpendicular to DE; (constr.)

*therefore AF is perpendicular to each of the straight lines GH, DE;*

but if a straight line stands at right angles to each of two straight lines in the point of their intersection, it shall also be at right angles to the plane passing through them; (XI. 4.)

but the plane passing through ED, GH is the plane BH;

**therefore AF is perpendicular to the plane BH.**

Wherefore, from the given point A, above the plane BH, the straight line AF is drawn perpendicular to that plane. Q. E. F.

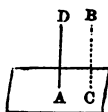
#### PROP. XII.—PROBLEM.

*To erect a straight line at right angles to a given plane, from a point given in the plane.*

(References—Prop. I. 31; XI. 8, 11.)

Let A be the point given in the plane.

It is required to erect a straight line from the point A at right angles to the plane.



## CONSTRUCTION

From any point B above the plane draw BC perpendicular to it ;  
(xi. 11.)

and from A draw AD parallel to BC. (i. 31.)

**Then AD shall be at right angles to the given plane.**

## DEMONSTRATION

Because AD, CB are two parallel straight lines, and one of them BC is at right angles to the given plane,

**therefore the other AD is also at right angles to it.**  
(xi. 8.)

Wherefore a straight line has been erected at right angles to a given plane from a point given in it. Q. E. F.

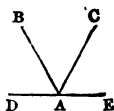
## PROP. XIII.—THEOREM.

*From the same point in a given plane, there cannot be two straight lines at right angles to the plane, upon the same side of it ;*

*and there can be but one perpendicular to a plane from a point above the plane.*

(References—Prop. xi. 3, 6, def. 3.)

For, if it be possible, let the two straight lines AB, AC be at right angles to a given plane from the same point A in the plane, and upon the same side of it.



## CONSTRUCTION

Let a plane pass through BA, AC ;

the common section of this with the given plane is a straight line passing through A ; (xi. 3.)

let DAE be their common section.

## DEMONSTRATION

Then the straight lines AB, AC, DAE must be in one plane ;  
and because CA is assumed to be at right angles to the given plane,

*therefore it must make right angles with every straight line meeting it in that plane ; (xi. def. 3.)*

but DAE, which is assumed to be in that plane, meets CA ;

*therefore CAE must be a right angle ;*

for the same reason BAE must be a right angle ;

*therefore the angle CAE must be equal to the angle BAE ; (i. ax. 11.)*

*and they must be in one plane ;*

which is impossible.

Also, from a point above a plane, there can be but one perpendicular to that plane ;

for, if there could be two, they would be parallel to one another, (xi. 6.)

which is absurd.

Therefore, from the same point, &c.

Q. E. D.

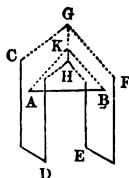
#### PROP. XIV.—THEOREM.

*Planes to which the same straight line is perpendicular, are parallel to one another.*

(References—Prop. I. 17 ; xi. def. 3, 8.)

Let the straight line AB be perpendicular to each of the planes CD, EF.

**Then the planes CD, EF shall be parallel to one another.**



#### CONSTRUCTION

If not, they shall meet one another when produced ; let them meet, and their common section will be a straight line GH ;

in GH take any point K, and join AK, BK.

#### DEMONSTRATION

Then, because AB is perpendicular to the plane EF, it must be perpendicular to the straight line BK which is in that plane ; (xi. def. 3.)

*therefore ABK must be a right angle.*

For the same reason, BAK must be a right angle;

*therefore the two angles ABK, BAK of the triangle ABK must be equal to two right angles;*

which is impossible; (i. 17.)

*therefore the planes CD, EF, though produced, do not meet;*

**that is, they are parallel to one another.** (xl. def. 8.)

Therefore, planes, &c.

Q. E. D.

### PROP. XV.—THEOREM.

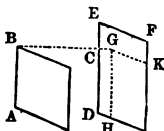
*If two straight lines meeting one another, be parallel to two straight lines which meet one another, but are not in the same plane with the first two;*

*then the plane which passes through these is parallel to the plane passing through the others.*

(References—Prop. i. 29, 31; xl. 4, 9, 11, 14, def. 3.)

Let AB, BC, two straight lines meeting one another, be parallel to DE, EF that meet one another, but are not in the same plane with AB, BC.

**Then the plane through AB, BC, and DE, EF shall be parallel to the plane through DE, EF.**



### CONSTRUCTION

From the point B draw BG perpendicular to the plane which passes through DE, EF, (xi. 11.)

and let it meet that plane in G;

through G draw GH parallel to ED, and GK parallel to EF. (i. 31.)

### DEMONSTRATION

And because BG is perpendicular to the plane through DE, EF,

*it makes right angles with every straight line meeting it in that plane;*  
(xi. def. 3.)

but the straight lines GH, GK in that plane meet it;

*therefore each of the angles BGH, BGK is a right angle.*

And because BA is parallel to GH, for each of them is parallel to DE, and they are not both in the same plane with it, (XI. 9.)

*therefore the angles GBA, BGH are together equal to two right angles ;*  
(I. 29.)

and BGH is a right angle ;

*therefore also GBA is a right angle, and GB perpendicular to BA ;*

for the same reason, GB is perpendicular to BC.

Since, then, the straight line GB stands at right angles to the two straight lines BA, BC, that cut one another in B ;

*therefore GB is perpendicular to the plane through BA, BC ;* (XI. 4.)

and it is perpendicular to the plane through DE, EF ; (constr.)

*therefore BG is perpendicular to each of the planes through AB, BC, and DE, EF ;*

but planes to which the same straight line is perpendicular, are parallel to one another ; (XI. 14.)

**therefore the plane through AB, BC is parallel to the plane through DE, EF.**

Wherefore, if two straight lines, &c.

Q. E. D.

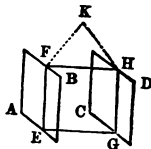
#### PROP. XVI.—THEOREM.

*If two parallel planes be cut by another plane, their common sections with it are parallels.*

(References—Prop. XI. 1.)

Let the parallel planes AB, CD be cut by the plane EFHG, and let their common sections with it be EF, GH.

**Then EF shall be parallel to GH.**



#### CONSTRUCTION

For if they be not parallel, EF, GH shall meet, if produced, either on the side of FH or EG.

*First, let them be produced on the side of FH, and meet in the point K.*

## DEMONSTRATION

Then, since  $EFK$  is assumed to be in the plane  $AB$ , every point in  $EFK$  must be in that plane ; (XI. 1.)

and  $K$  is a point in  $EFK$  ;

*therefore  $K$  must be in the plane  $AB$  ;*

for the same reason,  $K$  must be also in the plane  $CD$  ;

*wherefore the planes  $AB$ ,  $CD$  produced, would meet one another ;*

but they do not meet, since they are parallel by the hypothesis ;

*therefore the straight lines  $EF$ ,  $GH$  do not meet when produced on the side of  $FH$ .*

In the same manner it may be proved, that  $EF$ ,  $GH$  do not meet when produced on the side of  $EG$  ;

but straight lines which are in the same plane, and do not meet, though produced either way, are parallel ;

**therefore  $EF$  is parallel to  $GH$ .**

Wherefore, if two parallel planes, &c.

Q. E. D.

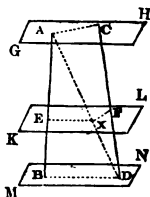
## PROP. XVII.—THEOREM.

*If two straight lines be cut by parallel planes, they shall be cut in the same ratio.*

(References—Prop. v. 11 ; VI. 2 ; XI. 16.)

Let the straight lines  $AB$ ,  $CD$  be cut by the parallel planes  $GH$ ,  $KL$ ,  $MN$ , in the points  $A$ ,  $E$ ,  $B$ , and in  $C$ ,  $F$ ,  $D$ .

**Then as  $AE$  is to  $EB$ , so shall  $CF$  be to  $FD$ .**



## CONSTRUCTION

Join  $AC$ ,  $BD$ ,  $AD$ , and let  $AD$  meet the plane  $KL$  in the point  $X$  ;  
and join  $EX$ ,  $XF$ .



and consequently it is perpendicular to CE ;

*wherefore ABF is a right angle ;*

but GFB is likewise a right angle ; (constr.)

*therefore AB is parallel to FG ; (I. 28.)*

and AB is at right angles to the plane CK ;

*therefore FG is also at right angles to the same plane. (XI. 8.)*

But one plane is at right angles to another plane when the straight lines drawn in one of the planes, at right angles to their common section, are also at right angles to the other plane ; (XI. def. 4.)

and any straight line FG in the plane DE which is at right angles to CE, the common section of the planes, has been proved to be perpendicular to the other plane CK ;

**therefore the plane DE is at right angles to the plane CK.**

In like manner, it may be proved that all planes which pass through AB are at right angles to the plane CK.

Therefore, if a straight line, &c.

Q. E. D.

#### PROP. XIX.—THEOREM.

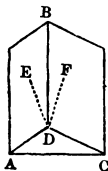
*If two planes which cut one another be each of them perpendicular to a third plane ;*

*then their common section shall be perpendicular to the same plane.*

(References—Prop. I. 11 ; XI. 13, def. 4.)

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common section of the first two.

**Then BD shall be perpendicular to the third plane.**



#### CONSTRUCTION

If it be not, from the point D draw, in the plane AB, the straight line DE at right angles to AD, the common section of the plane AB with the third plane ; (I. 11.)

and in the plane BC draw DF at right angles to CD, the common section of the plane BC with the third plane.



## DEMONSTRATION

Because the plane AB is perpendicular to the third plane, (hyp.)  
and DE is assumed to be drawn in the plane AB at right angles to AD,  
their common section,

*therefore DE must be perpendicular to the third plane.* (XI. def. 4.)

In the same manner, it may be proved, that DF must be perpendicular  
to the third plane,

*therefore, from the point D two straight lines must stand at right angles  
to the third plane, upon the same side of it, which is impossible ;*  
(XI. 13.)

therefore, from the point D there cannot be any straight line at right  
angles to the third plane, except BD the common section of the  
planes AB, BC ;

**therefore BD is perpendicular to the third plane.**

Wherefore, if two planes, &c.

Q. E. D.

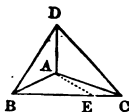
## PROP. XX.—THEOREM.

*If a solid angle be contained by three plane angles, any two of them are  
greater than the third.*

(References—Prop. I. 4, 20, 23, 25.)

Let the solid angle at A be contained by the three plane angles BAC,  
CAD, DAB.

**Then any two of them shall be greater than the third.**



## CONSTRUCTION

If the angles BAC, CAD, DAB be all equal, it is evident, that any two  
of them are greater than the third;

but if they are not, let BAC be that angle which is not less than either  
of the other two, and is greater than one of them DAB ;

and at the point A in the straight line AB, in the plane which passes  
through BA, AC, make the angle BAE equal to the angle DAB ;  
(I. 23.)

make AE equal to AD, and through E draw BEC cutting AB, AC in  
the points B, C, and join DB, DC.

## DEMONSTRATION

Then, because in the triangles BAD, BAE,  
 the side DA is assumed to be equal to AE, and BA is common,  
 the two DA, AB must be equal to the two EA, AB each to each ;  
 and the angle DAB is assumed to be equal to the angle EAB ;  
*therefore the base DB must be equal to the base BE ; (I. 4.)*  
 and because BD, DC are greater than CB ; (I. 20.)  
 and one of them BD has been proved equal to BE a part of CB,  
*therefore the other DC must be greater than the remaining part EC.*  
 (I. ax. 5.)

And because in the two triangles DAC, EAC,  
 AD is assumed to be equal AE, and AC is common,  
 but the base DC greater than the base EC ;  
*therefore the angle DAC must be greater than the angle EAC ; (I. 25.)*  
 and, by construction, the angle DAB is equal to the angle BAE ;  
*therefore the angles DAB, DAC must be together greater than BAE,*  
*EAC, that is, than the angle BAC, (I. ax. 4.)*  
 but BAC is not less than either of the angles DAB, DAC  
**therefore BAC, with either of them, is greater than the**  
**other.**

Wherefore, if a solid angle, &c.

Q. E. D.

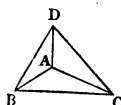
## PROP. XXI.—THEOREM.

*Every solid angle is contained by plane angles, which together are less than four right angles.*

(References—Prop. I. 32, cor. 1 ; XI. 20.)

First, let the solid angle at A be contained by three plane angles BAC, CAD, DAB.

**Then these three together shall be less than four right angles.**



## CONSTRUCTION

Take in each of the straight lines AB, AC, AD, any points B, C, D,  
 and join BC, CD, DB.

## DEMONSTRATION

Then, because the solid angle at B is contained by the three plane angles CBA, ABD, DBC, any two of them are greater than the third ; (XI. 20.)

*therefore the angles CBA, ABD are greater than the angle DBC ;*

for the same reason, the angles BCA, ACD are greater than the angle DCB, and the angles CDA, ADB, greater than BDC ;

*wherefore the six angles CBA, ABD, BCA, ACD, CDA, ADB, are greater than the three angles DBC, BCD, CDB ;*

but the three angles DBC, BCD, CDB are equal to two right angles ; (I. 32.)

*therefore the six angles CBA, ABD, BCA, ACD, CDA, ADB are greater than two right angles.*

And because the three angles of each of the triangles ABC, ACD, ADB are equal to two right angles,

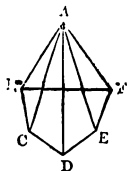
*therefore the nine angles of these three triangles. viz. the angles CBA, BAC, ACB, ACD, CDA, DAC, ADB, DBA, BAD, are equal to six right angles ;*

and of these six angles CBA, ACB, ACD, CDA, ADB, DBA are greater than two right angles ;

**therefore the remaining three angles BAC, CAD, DAB, which contain the solid angle at A, are less than four right angles.**

*Next, let the solid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB.*

**These shall be together less than four right angles.**



## CONSTRUCTION

Let the planes in which the angles are, be cut by a plane, and let the common sections of it with those planes be BC, CD, DE, EF, FB.

## DEMONSTRATION

Then, because the solid angle at B is contained by three plane angles CBA, ABF, FBC, of which any two are greater than the third, (XI. 20.)

*therefore the angles CBA, ABF, are greater than the angle FBC ;*

for the same reason, the two plane angles at each of the points C, D, E, F, viz. those angles which are at the bases of the triangles, having the common vertex A, are greater than the third angle at the same point, which is one of the angles of the polygon BCDEF ;

*therefore all the angles at the bases of the triangles are together greater than all the angles of the polygon.*

And because all the angles of the triangles are together equal to twice as many right angles as there are triangles, that is, as there are sides in the polygon BCDEF ; (I. 32.)

and that all the angles of the polygon, together with four right angles, are likewise equal to twice as many right angles as there are sides in the polygon ; (I. 32, cor. 1.)

*therefore all the angles of the triangles are equal to all the angles of the polygon together with four right angles ; (I. ax. 1.)*

but all the angles at the bases of the triangles are greater than all the angles of the polygon, as has been proved ;

**therefore the remaining angles of the triangles, viz. those of the vertex, which contain the solid angle at A, are less than four right angles.**

Wherefore, every solid angle, &c.

Q. E. D.

## BOOK XII.

## LEMMA.

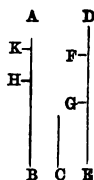
Which is the first proposition of the Tenth Book, and is necessary to some of the propositions of this Book.

*If from the greater of two unequal magnitudes, there be taken more than its half, and from the remainder more than its half; and so on; then there shall at length remain a magnitude less than the least of the proposed magnitudes.*

Let AB and C be two unequal magnitudes, of which AB is the greater.

If from AB there be taken more than its half, and from the remainder more than its half, and so on;

**Then there shall at length remain a magnitude less than C.**



## CONSTRUCTION

For C may be multiplied so as at length to become greater than AB; let it be so multiplied, and let DE its multiple be greater than AB, and let DE be divided into DF, FG, GE, each equal to C.

From AB take BH greater than its half, and from the remainder AH take HK greater than its half, and so on, until there be as many divisions in AB as there are in DE;

*and let the divisions in AB be AK, KH, HB, and the divisions in DE be DF, FG, GE.*

## DEMONSTRATION

Then, because DE is greater than AB,  
and that EG taken from DE is not greater than its half,  
but BH taken from AB is greater than its half ;  
*therefore the remainder GD is greater than the remainder HA.*

Again, because GD is greater than HA,  
and that GF is not greater than the half of GD,  
but HK is greater than the half of HA ;

*therefore the remainder FD is greater than the remainder AK ;*  
and FD is equal to C,

**therefore C is greater than AK ; that is, AK is less than C.** Q. E. D.

And if only the halves be taken away, the same thing may in the same way be demonstrated.

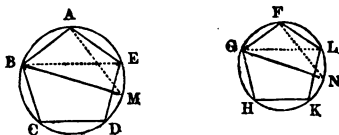
## PROP. I.—THEOREM.

*Similar polygons inscribed in circles, are to one another as the squares of their diameters.*

(References—Prop. III. 21, 31 ; v. 22, def. 10 ;  
VI. 4, 6, 20, def. 1.)

Let ABCDE, FGHLK be two circles, and in them the similar polygons ABCDE, FGHLK ; and let BM, GN be the diameters of the circles.

**Then as the polygon ABCDE is to the polygon FGHLK, so shall the square of BM be to the square of GN.**



## CONSTRUCTION

Join BE, AM, GL, FN.

## DEMONSTRATION

Then because the polygon ABCDE is similar to the polygon FGHLK,  
*therefore the angle BAE is equal to the angle GFL,*  
and as BA to AE, so is GF to FL ;

therefore the two triangles BAE, GFL having one angle in one equal to one angle in the other, and the sides about the equal angles proportionals, are equiangular;

*therefore the angle AEB is equal to the angle FLG ;*

but AEB is equal to AMB, because they stand upon the same circumference ; (III. 21.)

and the angle FLG is, for the same reason, equal to the angle FNG ;

*therefore also the angle AMB is equal to FNG.*

And the right angle BAM is equal to the right angle GFN ; (III. 31 )

*therefore the remaining angles in the triangles ABM, FGN are equal, and they are equiangular to one another ;*

therefore as BM to GN, so is BA to GF ; (VI. 4.)

*wherefore the duplicate ratio of BM to GN, is the same with the duplicate ratio of BA to GF ; (v. def. 10. and v. 22.)*

but the ratio of the square on BM to the square on GN, is the duplicate ratio of that which BM has to GN ; (VI. 20.)

and the ratio of the polygon ABCDE to the polygon FGHKL is the duplicate of that which BA has to GF ; (VI. 20.)

**therefore as the polygon ABCDE is to the polygon FGHKL, so is the square of BM to the square of GN.**

Wherefore, similar polygons, &c.

Q. E. D.

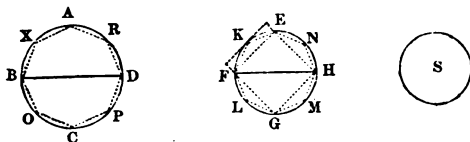
### PROP. II.—THEOREM.

*Circles are to one another as the squares of their diameters.*

(References—Prop. I. 41 ; IV. 6 ; VII. 14 ; XII. 1.)

Let ABCD, EFGH be two circles, and BD, FH their diameters.

**Then as the square of BD to the square of FH, so shall the circle ABCD be to the circle EFGH.**



For, if not, the square of BD must be to the square of FH, as the circle ABCD is to some space either less than the circle EFGH, or greater than it.

First, if possible, let it be as the circle ABCD is to a space S less than a circle EFGH.

In the circle EFGH inscribe the square EFGH. (IV. 6.)

This square is greater than half of the circle EFGH.

Because, if through the points E, F, G, H, there be drawn tangents to the circle, the square EFGH is half of the square described about the circle ; (I. 41.)

and the circle is less than the square described about it ;

*therefore the square EFGH is greater than half the circle.*

Divide the circumferences EF, FG, GH, HE, each into two equal parts in the points K, L, M, N, and join EK, KF, FL, LG, GM, HM, HN, NE ;

*then each of the triangles EKF, FLG, GMH, HNE, is greater than half of the segment of the circle in which it stands ;*

for, if straight lines touching the circle be drawn through the points K, L, M, N, and the parallelograms upon the straight lines EF, FG, GH, HE be completed, each of the triangles EKF, FLG, GMH, HNE is the half of the parallelogram in which it is ; (I. 41.)

but every segment is less than the parallelogram in it is ;

*therefore each of the triangles EKF, FLG, GMH, HNE is greater than half the segment of the circle which contains it.*

Again, if the remaining circumferences be divided each into two equal parts, and their extremities be joined by straight lines, by continuing to do this, there will at length remain segments of the circle, which together are less than the excess of the circle EFGH above the space S ;

because, by the preceding Lemma, if from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and so on, there will at length remain a magnitude less than the least of the proposed magnitudes.

Let then the segments EK, KF, FL, LG, GM, MH, HN, NE be those that remain, and are together less than the excess of the circle EFGH above S ;

*therefore the rest of the circle, viz. the polygon EKFLGMHN, is greater than the space S.*

Describe likewise in the circle ABCD the polygon AXBOCPDR similar to the polygon EKFLGMHN ;

*therefore as the square on BD is to the square on FH, so is the polygon AXBOCPDR to the polygon EKFLGMHN ; (XII. 1.)*



but the square on BD is also to the square on FH, as the circle ABCD is to the space S ; (hyp.)

*therefore as the circle ABCD is to the space S, so must the polygon AXBOCPDR be to the polygon EKFLGMHN ; (v. 11.)*

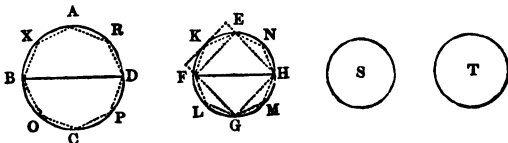
but the circle ABCD is greater than the polygon contained in it ;

*therefore the space S must be greater than the polygon EKFLGMHN.* (v. 14.)

But it is likewise less, as has been demonstrated ; which is impossible ;  
*therefore the square of BD is not to the square of FH, as the circle ABCD is to any space less than the circle EFGH.*

In the same manner, it may be demonstrated, that neither is the square of FH to the square of BD, as the circle EFGH is to any space less than the circle ABCD.

Nor is the square of BD to the square of FH, as the circle ABCD is to any space greater than the circle EFGH ;



For, if possible, let it be so to T, a space greater than the circle EFGH ;  
*then, inversely, as the square of FH to the square of BD, so must the space T be to the circle ABCD ;*

but as the space T is to the circle ABCD, so must the circle EFGH be to some space, which is less than the circle ABCD, because the space T is assumed to be greater than the circle EFGH ; (v. 14.)

*therefore as the square of FH is to the square of BD, so must the circle EFGH be to a space less than the circle ABCD,*

which has been demonstrated to be impossible ;

*therefore the square of BD is not to the square of FH as the circle ABCD is to any space greater than the circle EFGH.*

And it has been demonstrated, that neither is the square of BD to the square of FH, as the circle ABCD to any space less than the circle EFGH ;

**wherefore, as the square on BD is to the square on FH, so is the circle ABCD to the circle EFGH.**

Therefore, circles, &c.

Q. E. D.

# NOTES

## ON

### EUCLID'S ELEMENTS.

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EUCLID, the author of the celebrated 'Elements' which go by his name, lived at Alexandria in the time of the first Ptolemy, B.C. 323-283, and is considered to have been the founder of the Alexandrian Mathematical School. Since the time when the Elements were first used in that school, they have been adopted as the basis of mathematical instruction in all civilised countries, up to the present day. 'It is a remarkable fact in the history of science,' says Professor Playfair, 'that the oldest book of Elementary Geometry is still considered to be the best, and that the writings of Euclid, at the distance of 2,000 years, continue to form the most approved introduction to the mathematical sciences.' Many other treatises on Geometry have been written, but the work of Euclid has been almost universally preferred, not only for the elegance and strictness of its reasoning, but also for the admirable manner in which it is arranged for the purpose of instruction. We must not, however, regard Euclid as the author or discoverer of all the propositions contained in the Elements. Many of them, it is certain, were known before his time, for numerous philosophers of celebrity had laboured in the same field before him. But as the compiler and arranger of the work, the great Greek Geometer is entitled to the highest praise.

The Elements, as left by Euclid, consist of fifteen books, of which the first six, the eleventh, and twelfth, form the most valuable part, and, accordingly, few editions contain more than these. The seventh, eighth, and ninth books treat of number, the tenth of commensurable and incommensurable magnitudes, and the last five of solids. Two of the fifteen books are believed by some not to have been written by Euclid, but to have been added by Hypsicles.

The edition of Euclid mostly used in this country is that of Dr. Robert Simson, which has, in some form or other, superseded most others. This edition, which has in the main been followed in the present work, presents to us the principal books of the original freed from those errors which had crept into it while passing through the hands of the ancient editors. While aiming to restore the text to its original purity, Dr. Simson has in many cases removed imperfections that probably belonged to the original work.

The most exact translation of the Elements is that by the Rev. James Williamson, Fellow of Hertford College, Oxford, published in two volumes,

quarto, the first at Oxford in 1781, and the second in London, in 1783. Editions of the fifteen books have also been published by Barrow, Gregory, Peyrard and several others. The article *Euclides* in Dr. Smith's 'Dictionary of Greek and Roman Biography,' written by Professor De Morgan, contains an account of the principal editions of Euclid which have been published since the revival of learning in Europe, and to this article the student is referred.

## ON THE FIRST BOOK.

### Definitions.

The entire system of Geometry, as a science, is based on a few definitions and axioms. The first seven definitions, it is said, are defective, and many substitutes have therefore been proposed for them. Into this controversy it would lead us beyond our present limits to enter. It is sufficient to remark that there is considerable difficulty in framing accurate definitions of the simple and primitive ideas of which geometry treats; and it is scarcely possible to do so in a manner which would satisfy every mind.

In the first definition, a *point* is stated to be 'that which has no parts, or which has no magnitude,' which may be regarded as a caution, that in determining a point we should take no notice of its size. It simply indicates *position*; and as we cannot represent such an object to the eye, it is sufficiently near the truth to call a point *that which has an infinitely small extension*. A *line* is defined to be 'length without breadth;' for when its length is ascertained, we know all that is necessary concerning it. The breadth forms no part of its conception, and does not enter into geometrical reasoning. A line is thus, in geometry, extension in one direction only, and it might be defined as *the path described by a point moving only in one direction*. A *superficies*, in the same way, is said to be 'that which has only length and breadth,' or to be extension in two directions.

To understand rightly the definitions of a point, line, and superficies, it is necessary, says Simson, to consider a solid, that is a magnitude which has length, breadth, and thickness; for these all arise from a solid and exist in it. 'The boundary or boundaries which contain a solid are called superficies; or the boundary which is common to two solids which are contiguous, or which divides one solid into two contiguous parts, is called a superficies.' 'The boundary of a superficies is called a line, or a line is the common boundary of two superficies that are contiguous, or which divides one superficies into two contiguous parts.' 'The boundary of a line is called a point; or a point is the common boundary or extremity of two lines that are contiguous.'

It is important to remember that in the ninth definition the magnitude of an angle is determined by the divergence, and not the length, of the lines which include it. The sides or legs of an angle may be increased or diminished in length without affecting its magnitude. An angle is in fact a distinct species of magnitude, and its equality may be proved, like that of triangles and other figures, by superposition. 'The angle made by a straight line with its own continuation is a definite angular magnitude, and its half is the best definition of a right angle. It is to be regretted that there is no single phrase for two right angles.'—*De Morgan*.

According to the fifteenth definition, the *circle* is the space enclosed, and the *circumference* is the line that bounds it. In the third book, the *circumference* is often called the circle.

The definition of a square is redundant, inasmuch as it would have been sufficient to mention one right angle; for if a quadrilateral have all its sides equal, and have one right angle, it may be proved, as in I. 46, that it will have all its angles right angles. Euclid's definition has, however, the advantage of being clear and concise.

### Postulates.

There are certain operations, the possibility of performing which must be conceded, which are necessary to a demonstration. These are called the Postulates, or Problems granted without construction. They permit the use of the rule to draw straight lines, and the compasses to describe circles. The rule, it should be observed, must not be graduated, nor the ends touched, so that we may not draw lines of a given length. And the compasses are not to be used for any other purpose than to describe a circle from a given point, with a given radius; that is, they must close the moment they are taken off the paper.

### Axioms.

There are certain general principles or self-evident truths, on which geometrical, like all other reasoning, must be based. These are the Axioms, or Theorems, granted without demonstration. We must not, it must be remarked, admit among the number of the axioms any proposition, however self-evident, which can be deduced from axioms already granted. And, on the other hand, if a proposition be not self-evident, and yet is not capable of being demonstrated, we are compelled to consider it as an axiom, as is the case with the twelfth axiom.

The eighth axiom is the definition of geometrical equality. Superposition, by which one figure is conceived to be placed upon another in such a manner that all their parts coincide, or exactly fill the same space, enables us to prove that the two figures are exactly equal to each other.

### The Propositions.

Propositions are of two kinds: those requiring something to be done, which are called *problems*; and those requiring something to be proved, which are called *theorems*. Every proposition consists of two parts: a problem consists of the *data*, or things granted, and the *quæsitæ*, or things demanded; a theorem consists of the *hypothesis*, or the things supposed, and the *predicate*, or things to be demonstrated. It is of the greatest importance that the learner should clearly discriminate in each case between these two parts of a proposition, for it would be useless for anyone to enter upon the demonstration until he understands what is assumed or granted, and what is required to be done or proved.

According to Proclus, a complete proposition consists of six parts:—1. The *general enunciation* of the problem or theorem. 2. The *particular enunciation*, which refers to the figure in question. 3. The *determination*, which announces the thing sought. 4. The *construction*, which states the necessary parts of the figure to be drawn. 5. The *demonstration*, which proves that the theorem is true or false, or that the problem is solved. 6. The *conclusion*, which is a mere repetition of the enunciation.

The *First Book* is devoted chiefly to the properties of triangles and parallelograms.

A triangle is the most simple of all rectilinear figures, since less than three straight lines cannot form any figure. All other rectilinear figures may be resolved into it by drawing straight lines from any point within them to their several angles. The triangle is therefore, in effect, the element of all rectilinear figures; and on its properties, the properties of all other rectilinear figures depend.

Next to the triangle, the most important rectilinear figure is the parallelogram, which involves the doctrine of parallels, with which a considerable portion of the first book is also taken up.

I. 2. It is a useful exercise to vary the position which the given straight line and given point may have with respect to each other. For instance, the given point may be joined with either extremity of the given line, the equilateral triangle may be constructed on either side of this line, and the sides of the equilateral triangle may be produced either way.

This proposition shows the necessity for the caution mentioned in the Postulates with regard to the use of the compasses. For as the compasses are considered to close as soon as they are removed from the paper, we are prevented from measuring the required line by this instrument.

I. 4. This is the first case of the equality of triangles. This proposition is proved by the method called Superposition.

I. 6 is the converse of one part of I. 5, that is, the hypothesis of this is the predicate of the fifth, and the predicate of this is the hypothesis of the fifth.

The demonstration of this proposition is *indirect*, a mode of proof which establishes a result by showing the absurdity of supposing it to be untrue. The *argumentum ad absurdum* is well adapted to converse truths, as in this instance.

I. 7 seems to have been introduced simply for the sake of proving the eighth. The argument employed in this demonstration, is that called a *dilemma*.

I. 8 is the second case of the equality of triangles.

I. 9. The equilateral triangle is constructed on the side remote from A, or it might happen that the vertex F might coincide with A, in which case there would be no solution of the problem. By this proposition we are enabled to divide an angle into any number of equal parts expressed by a power of two, as, 4, 8, 16, &c.

I. 11. The corollary to this proposition was added by Simson. It is considered objectionable, and is often omitted.

I. 12. It is necessary that the straight line AB be of unlimited length, or it might perhaps happen that the circle might not cut it in two points.

I. 13. 'The words "makes angles with it," are introduced to exclude the case in which the line AB is at the extremity of DC.'—*Lardner*.

I. 14. The words 'upon the opposite sides of it' are important; for if they were omitted, the two straight lines might form angles together equal to two right angles, and yet not lie in the same straight line.

I. 17 is the converse of axiom 12.

I. 16. The order in this and the following proposition corresponds to I. 5. and I. 6. The difficulty of distinguishing between these two propositions arises from the *hypothesis* being stated before the *predicate*.

I. 20. This proposition has been ridiculed as being 'evident even to asses.'

Though self-evident, it is demonstrated, not to *convince* the student of its truth, but to avoid increasing the number of the axioms.

**I. 24.** The limitation that DE is to be the side which is not the greater of the two, was added by Simson, to render the proof generally applicable. Without these words, there would arise *three* different cases. The point G might fall *on, above, or below* the base EG.

**I. 26.** This forms the *third* case of the equality or identity of two triangles. The first and second cases were contained in **I. 4.** and **I. 8.** The three combined will give rise to the four following cases:—

- a. Given two sides and the included angle. (**I. 4.**)
- b. Given the three sides. (**I. 8.**)
- c. Given two angles and an adjacent side. (**I. 26.**)
- d. Given two angles and a side opposite to the equal angles. (**I. 26.**)

There is also one other case, which, however, is of no great use, viz.—

- e. Given two sides, and an angle opposite to one side of one triangle, equal to the angle opposite to the equal side of the other triangle.

This is called the *ambiguous case*, since it is true only under certain conditions, viz. that the opposite angles must be of the same kind; that is, both right angles, both acute, or both obtuse.

**I. 27.** Scarcely anything is more obvious to the senses, and yet, at the same time, more difficult to prove, than the properties of parallel lines. Euclid gets over the difficulty by assuming the following as an axiom:—‘If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles; these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.’ This, it has been objected, is not self-evident, and ought therefore to have been matter of demonstration. To supersede the necessity of assuming this axiom, numerous other theories have been proposed. Most of these, however, have the defect of being very complicated and subtle, and therefore quite unfit for beginners; while Euclid’s, after having once admitted the twelfth axiom, is simple and clear.

**I. 32.** The two corollaries were added by Simson. There is another kind of exterior angle, besides that to which Euclid confines himself. When any of the sides bend inwards, they form what are called *re-entrant* angles. The second corollary is not true for a figure with one or more of these angles.

**I. 35.** Simson has here departed from the original, in order to avoid the three cases which Euclid’s method requires. The word *equal* in this proposition assumes a new meaning. The equality is that of *area*, and not of identity of sides and angles.

**I. 36.** It is supposed that the equal bases are placed in the same straight line.

**I. 44.** A parallelogram is said to be *applied* to a straight line, when it has that line for one of its sides.

**I. 47.** This celebrated proposition is said to have been discovered by Pythagoras. The side opposite to the right-angle is called the *hypotenuse*. In the diagram the squares are described on the *outer* sides of the triangle; but the proposition may be demonstrated when the squares are on the *inner* sides of the triangle, or some on one side and some on the other.

## A CLASSIFICATION OF THE PROPOSITIONS OF BOOK I.

**Problems.***A. Relating to Straight Lines.*

1. From a given point, to draw a straight line equal to a given finite straight line . . . . . I. 2.
2. From the greater of two given straight lines to cut off a part equal to the less . . . . . I. 3.
3. To bisect a given finite straight line . . . . . I. 10.
4. To draw a straight line at right angles to a given straight line, from a given point in the same . . . . . I. 11.
5. To draw a straight line perpendicular to a given straight line of unlimited length, from a given point in the same . . . . . I. 12.
6. Through a given point to draw a straight line parallel to a given straight line . . . . . I. 31.

*B. Relating to Rectilineal Angles.*

1. At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle . . . . . I. 23.
2. To bisect a given rectilineal angle . . . . . I. 9.

*C. Relating to Triangles.*

1. To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third . . . . . I. 22.
2. To describe an equilateral triangle upon a given finite straight line . . . . . I. 1.

*D. Relating to Parallelograms.*

1. To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle . . . . . I. 42.
2. To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle . . . . . I. 44.
3. To describe a parallelogram equal to a rectilineal figure, and having an angle equal to a given rectilineal angle . . . . . I. 45.
4. To describe a square upon a given straight line . . . . . I. 46.

**Theorems.***A. Relating to Angles formed by the Meeting or Intersection of Straight Lines.*

1. The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles . . . . . I. 13.
2. If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles; then these two straight lines are in one and the same straight line . . . . . I. 14.
3. If two straight lines cut one another, the vertical or opposite angles are equal . . . . . I. 15.

*B. Relating to the Sides and Angles of Triangles.*

1. The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base are equal . . . . . I. 5.
2. Conversely, if two angles of a triangle be equal to each other; the sides also which subtend, or are opposite to the equal angles, are equal to one another . . . . . I. 6.
3. The greater side of every triangle is opposite to the greater angle . . . . . I. 18.
4. The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it \* . . . . . I. 19.
5. Any two sides of a triangle are together greater than the third . . . . . I. 20.
6. If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles . . . . . I. 16.
7. Any two angles of a triangle are together less than two right angles . . . . . I. 17.
8. If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles † . . . . . I. 32.
9. In every right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the square described upon the sides which contain the right angle . . . . . I. 47.
10. And conversely, if the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle . . . . . I. 48.

*C. Relating to the Comparison of Triangles as to Equality or Inequality.*

1. If two triangles be upon the same base, and on the same side of it, they cannot have their sides which are terminated in the one extremity of the base equal to one another, and likewise those which are terminated in the other extremity . . . . . I. 7.
2. Two triangles are equal, or of the same identity of form, when—
  - a. Two sides of the one are equal to two sides of the other, each to each, and the angles contained by these sides are equal to each other . . . . . I. 4.
  - b. Two sides and a base of the one are equal to two sides and a base of the other . . . . . I. 8.
  - c. Two angles in the one are respectively equal to two angles in the other, and the side adjacent to those two angles in the one, equal to the corresponding side in the other . . . . .
  - d. Two angles in the one are respectively equal to two angles in the other, and the side opposite to these equal angles in the one, equal to the corresponding side in the other . . . . . I. 26.
3. If two triangles have two sides of the one equal to two sides of the other, each to each, but the included angle of the one be greater

\* 'The four (i.e. 1, 5, 6, 18, 19) might be thus combined:—one angle of a triangle is greater or less than another, or equal to it, according as the side opposed to the one is greater or less than, or equal to the side opposed to, the other; and *vice versa*.'—*Lord Avebury*.

† The 32nd proposition thus includes the 16th and 17th.



than the included angle of the other, the base of that which is the greater angle is greater than the base of the other . . . . . I. 24.

4. If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of the one which has the greater base is greater than the angle contained by the sides equal to them of the other \* . . . . . I. 25.

5. Triangles are equal in area when—

a. They are upon the same base and between the same parallels . . . . . I. 37.

b. They are upon equal bases and between the same parallels . . . . . I. 38.

6. Conversely, triangles are between the same parallels when—

a. They are equal in area, and upon the same base, and on the same side of it . . . . . I. 39.

b. They are upon equal bases in the same straight line, and on the same side of it . . . . . I. 40.

#### D. Relating to Parallel Lines.

1. Two straight lines are parallel, when another line falling upon them makes—

a. The alternate angles equal to one another . . . . . I. 27.

b. The exterior angle equal to the interior and opposite upon the same side of the line . . . . . I. 28.

c. The interior angles upon the same side together equal to two right angles . . . . .

2. And conversely, if a straight line fall upon two parallel straight lines, it makes—

a. The alternate angles equal to one another . . . . .

b. The exterior angle equal to the interior and opposite upon the same side . . . . . I. 29.

c. And the two interior angles upon the same side together together equal to two right angles . . . . .

3. If two straight lines are parallel to the same straight line, they are parallel to each other . . . . . I. 30.

#### E. Relating to Parallelograms.

1. The opposite sides and angles of a parallelogram are equal to one another, and the diameter that bisects it, that is, divides it into two equal parts. . . . . I. 34.

2. Parallelograms are equal in area when—

a. They are upon the same base and between the same parallels . . . . . I. 35.

b. They are upon equal bases and between the same parallels . . . . . I. 36.

3. If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram is double of the triangle . . . . . I. 41.

\* 'The 4th, 8th, 24th, and 25th propositions may be announced together thus:—'If two triangles have two sides of the one respectively equal to two sides of the other, the remaining side of the one will be greater or less than, or equal to, the remaining side of the other, according as the angle opposed to it in the one is greater or less than or equal to the angle opposed to it in the other, or *vice versa*.'—Lardner.

4. The complements of the parallelograms which are about the diameters of any parallelograms are equal to one another . I. 43.

The following summary of the Propositions of Book I., by Professor De Morgan, will be useful to the student.

'Without describing minutely the contents of the First Book of the Elements, we may observe that there is an arrangement of the Propositions which will enable any teacher to divide it into sections. Thus Prop. 1—3 extend the power of construction to the drawing of a circle with any centre and any radius; 4—8 are the basis of the theory of equal triangles; 9—12 increase the power of construction; 13—15 are solely on relations of angles; 16—21 examine the relations of parts of one triangle; 22, 23 are additional constructions; 23—26 augment the doctrine of equal triangles; 27—31 contain the theory of parallels; 32 stands alone, and gives the relation between the angles of a triangle; 33, 34 give the first properties of a parallelogram; 35—41 consider parallelograms and triangles of equal areas, but different forms; 42—46 apply what precedes to augmenting power of construction; 47, 48 give the celebrated property of a right angled triangle and its converse.'—Article *Euclides*, in 'Smith's Dictionary of Greek and Roman Biography.'

### ON THE SECOND BOOK.

In Book I. the general properties of parallelograms were considered. In Book II. a particular kind of parallelogram, viz. the *rectangle*, is considered, and under an entirely new aspect. To determine a rectangle or right-angled parallelogram, it is only necessary to know two of the sides which contain one of the right angles. In this book, therefore, a rectangle is expressed by its two conterminous sides.

In the Appendix it is shown how the propositions of this book may be applied to *number*. It is only necessary to remark here, that the algebraical and arithmetical demonstrations there given must never be substituted as an equivalent for the geometrical. For it must be always remembered, that geometry treats only of *magnitude* or *space*, but not of *number*.

The first eight propositions are all based on the obvious truth, that the whole is equal to all its parts together.

II. 2 and II. 3 might be thus combined. 'The difference between the rectangle under two lines and the square of one of them is the rectangle under that one and their difference. If that one be greater, this is the second proposition; and if it be less, it is the third.'—*Lardner*.

II. 6 is identical with II. 5; it differs only in appearance. They may be both included under one statement; 'the rectangle under the sum and difference of two lines is equal to the difference of the squares on those lines.' In II. 5 the whole line is the sum of the unequal parts, and each of the equal parts is half the sum of the unequal parts. And since the greater of the unequal parts exceeds one of the equal parts by the intermediate part, and one of the equal parts exceeds the lesser part by the same intermediate part, it follows that the greater part exceeds the lesser by twice the intermediate parts. Therefore the intermediate part is half the difference of the unequal parts. In II. 6 the line AB is cut externally in the point D; the unequal parts are AD, DB, and the intermediate part is CD.

II. 7 may be joined to II. 4, thus 'the rectangle under the sum and difference of the squares on those lines.'

II. 10. This proposition is identical with the ninth, and bears the same relation to it as the sixth does to the fifth.

II. 12 and II. 13 should be viewed in connection with I. 47. They are all three of the greatest importance in Trigonometry.

II. 14 is a particular case of VI. 25.

## A CLASSIFICATION OF THE PROPOSITIONS OF BOOK II.

The second book, though small in bulk, is generally felt to be a difficult one by beginners. This difficulty, to some extent, proceeds from the many forms in which the same proposition is presented. It may perhaps assist the student in gaining a comprehensive view of the whole book, to look at the propositions as exhibiting the relations that exist between the rectangles formed by a line, and the various parts into which it can be divided.

### Theorems.

#### A. Relating to a Straight Line divided into any two parts.

Under this aspect there will be three lines to be considered: 1st, the whole line; 2nd, its greater part; 3rd, its less part.

1. The square on the whole line is equal to the rectangles under the whole and each of the parts . . . . . II. 2.
2. The rectangle under the whole and one of the parts, is equal to the rectangle under the two parts, together with the square on the aforesaid part . . . . . II. 3.
3. The square on the whole line is equal to the squares on the two parts, together with twice the rectangle under the parts . . . . . II. 4.
4. The squares on the whole line, and on one of the parts, are equal to twice the rectangle under the whole and that part, together with the square on the other part . . . . . II. 7.
5. Four times the rectangle under the whole line, and one of the parts, together with the square on the other part, is equal to the square on the straight line, which is made up of the whole and that part . . . . . II. 8.

#### B. Relating to a Straight Line, divided in two points, equally and unequally.

Under this aspect there will be several lines to be considered: 1st, the whole line; 2nd, the equal parts; 3rd, the unequal parts; 4th, the intermediate part or the part between the points of section.

1. The rectangle under the unequal parts, together with the square on the intermediate part, is equal to the square on half the line . . . . . II. 5.
2. If a straight line be bisected, and produced to any point, the rectangle under the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced . . . . . II. 6.
3. The squares on the two unequal parts are together double of the square on half the line, and of the square on the intermediate part . . . . . I. 9.
4. If a straight line be bisected and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line

bisected, and of the square on the line made up of the half and the part produced . . . . . II. 10.

C. *Relating to a Straight Line divided into any number of parts, and another Straight Line which is not divided.*

1. The rectangle under the two lines is equal to the rectangles under the undivided line, and the several parts of the divided line . . . . . II. 1.

D. *On the Relations between the Sides of a Triangle and the Perpendicular from the Vertex.*

1. In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle under the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle . . . . . II. 12.

2. In every triangle, the square on the side subtending either of the acute angles is less than the squares on the sides containing that angle, by twice the rectangle under either of these sides, and the straight line intercepted between the acute angle and the perpendicular let fall upon it from the opposite angle\* . . . . . II. 13.

# Problems.

1. To divide a given straight line into two parts, so that the rectangle under the whole and one of the parts is equal to the square on the other part . . . . . II. 11.

2. To describe a square that shall be equal to a given rectilineal angle . . . . . II. 14.

## ON THE THIRD BOOK.

In Book III. are demonstrated the properties of the circle, the most perfect of all figures. This book is considered of great importance, as the science of Mathematics is, to a great extent, founded upon its propositions.

Def. 1. Simson calls this a theorem, and not a definition. The equality of circles may be shown by superposition, that is, by placing one upon the other, so that the centre of one is on the centre of the other.

Many of the demonstrations of this book are *indirect*, a mode of proof quite as correct as the *direct*. Some of these indirect propositions, as the first, for example, are incapable of being proved directly.

III. 2. We have preferred the *direct* proof of Commandine to the *indirect* proof of Euclid, on account of the simplicity of the figure.

III. 5. 'This proposition may be better announced thus: Concentric circles cannot meet, and that which has the less radius will be included within the other.'—Lardner.

\* II. 12 and II. 13, taken in connection with I. 47, may be thus combined:—'The square of the base of a triangle is less than, equal to, or greater than the sum of the squares of the sides, according as the vertical angle is less than, equal to, or greater than a right angle.'

**III. 11 and III. 12.** In these propositions it is proved that the line which joins the centres of the two touching circles passes through a point of contact.

In **III. 13** it is proved that this is the only point of contact.

**III. 17.** It is evident that there can be drawn two tangents from the point A, lying in different sides to the centre. And these two tangents are equal in all respects.

**III. 20.** This demonstration is slightly defective. In the first case it is assumed that the sum of the doubles of two magnitudes is double of their sum; and in the third case it is assumed that the difference of the doubles of two magnitudes is double of their difference, both of which are afterwards proved in Book v.

This proposition is equally true of those circles which have their central angles greater than two right angles; that is, it may be proved by joining BF and EC in Fig. 1, that the reverse angle BEC is double of the angle BFC.

If this extension of the idea of an angle were adopted, some of the following demonstrations might be abridged.

**III. 21.** If the term angle be extended in the manner just mentioned, there would require no division of this proposition into two cases; for in case 2, if B, C, C, D, be joined, the reverse angle BCD at the centre—that is, the angle greater than two right angles—is double of each of the angles BAD, BED. Euclid, however, only gives the first case; the second has been added by Simson.

**III. 26—III. 29.** Whatever is proved of *equal* circles in these propositions is true also of the *same* circle.

**III. 35** is an illustration of the truth that a product remains constant, while the sum of its factors is greater or less according as they are more or less unequal.

#### A CLASSIFICATION OF THE PROPOSITIONS OF BOOK III.

##### Problems relating to Circles.

1. To find the centre of a given circle . . . . . **III. 1.**
2. To draw a straight line from a given point, either without or in the circumference, which shall be a tangent to a given circle . . . **III. 17.**
3. To bisect a given arc, that is, to divide it into two equal parts . **III. 30.**
4. A segment of a circle being given, to describe the circle of which it is the segment . . . . . **III. 25.**
5. Upon a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle . . **III. 33.**
6. From a given circle to cut off a segment which shall contain an angle equal to a given rectilineal angle . . . . . **III. 34.**

##### Theorems.

###### A. Relating to Straight Lines drawn from any Point to the Circumference of a Circle.

1. If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it cuts it at right angles; and conversely, if it cut it at right angles, it bisects it . . . . . **III. 3.**

in a circle two straight lines cut one another, which do not pass through the centre, they do not bisect each other . . . . . III. 4.

Two straight lines cut one another within a circle, the rectangle under the segments of one of them is equal to the rectangle under the segments of the other . . . . . III. 35.

From any point without a circle two straight lines be drawn, which cuts the circle, and the other touches it; the rectangle under the whole line which cuts the circle, and the part without the circle, is equal to the square on the line which touches it . . . . . III. 36.

And conversely, if from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets the circumference, the rectangle under the whole line which cuts the circle, and the part without the circle, be equal to the square on the line which meets the circle, the line which meets shall touch the circle . . . . . III. 37.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the nearer part of that diameter is the least; and, of the rest, that is nearer to the line which passes through the centre is always greater than the one more remote; and from the same point there can be drawn only two equal straight lines to the circumference, one on each side of the diameter . . . . . III. 7.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre, of those which fall upon the concave part of the circumference, the greatest is that which passes through the centre; and, of the rest, that which is nearer to the one passing through the centre is always greater than one more remote; but of those which fall upon the convex part of the circumference, the least is that which is nearer to the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote. And only two equal straight lines can be drawn from the same point to the circumference, one upon each side of the line which passes through the circumference . . . . . III. 8.

#### B. Relating to the Contact of Circles.

If one circumference of a circle cut another, they have not the same centre . . . . . III. 5.

If one circle touch another internally, they have not the same centre . . . . . III. 6.

One circumference of a circle cannot cut another in more than two points . . . . . III. 10.

If one circle touch another internally in any point, the straight line which joins their centres, being produced, passes through that point of contact . . . . . III. 11.

If two circles touch each other externally in any point, the straight line which joins their centres passes through that point of contact . . . . . III. 12.

6. One circle cannot touch another in more points than in one, whether it touches it on the inside or outside . . . . . III. 13.

### C. Relating to Tangents.

1. A straight line drawn from the extremity of the diameter of a circle perpendicular to it falls without the circle; and if any straight line be drawn from a point within that perpendicular to the point of contact, it cuts the circle . . . . . III. 16.
2. If a straight line be a tangent to a circle, the straight line drawn from the centre to the point of contact is perpendicular to it . . . . . III. 18.
3. If a straight line be a tangent to a circle, the straight line drawn perpendicular to it from the point of contact passes through the centre of the circle . . . . . III. 19.
4. If a straight line be a tangent to a circle, and from the point of contact a straight line be drawn cutting the circle, the angle made by this line with the tangent is equal to the angle in the alternate segment of the circle . . . . . III. 32.

### D. Relating to the Angles in a Circle.

1. The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference . . . . . III. 20.
2. The angles in the same segment of a circle are equal to one another . . . . . III. 21.
3. The opposite angles of any quadrilateral figure inscribed in a circle, are together equal to two right angles . . . . . III. 22.
4. In equal circles, equal angles stand upon equal arcs, whether the angles be at the centres or circumferences . . . . . III. 26.
5. In equal circles, the angles which stand upon equal arcs are equal to one another, whether they be at the centres or circumferences . . . . . III. 27.
6. In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle . . . . . III. 31.

### E. Relating to Segments and their Chords.

1. Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another . . . . . III. 23.
2. Similar segments of circles upon equal straight lines are equal to one another . . . . . III. 24.
3. Equal straight lines in a circle are equally distant from the centre; and conversely, those which are equally distant from the centre are equal to one another . . . . . III. 14.
4. The diameter is the greatest straight line in a circle; and of the rest, that which is nearer to the centre is always greater than one more remote; and conversely, the greater is nearer to the centre than the less . . . . . III. 15.

5. In equal circles, equal straight lines cut off equal arcs, the greater equal to the greater, and the less to the less . . . . . III. 28.  
 6. In equal circles, equal arcs are subtended by equal straight lines . . . . . III. 29.

# ON THE FOURTH BOOK.

The inscription and circumscription of triangles, squares, and polygons in and about circles, forms the subject of Book iv.

A polygon whose sides and angles are equal is called a *regular* polygon.

IV. 2. In this proposition Euclid assumes that the lines AB, AC will cut the circumference.

IV. 4. In this proposition it is assumed that the straight lines, BD, CD will meet in the point D. This may be easily proved, by showing that the angles they make with the side BC are together less than two right angles, after the manner of I. 44 and II. 10.

IV. 7. From this it is clear that the only rectangle which can be circumscribed about a circle is a square, but that either a square or a rectangle may be inscribed in it.

IV. 10. This proposition shows the means by which we may divide a right angle into five equal parts; for the vertical angle being the fifth part of two right angles, if we bisect it, we shall obtain the fifth part of a right angle.

IV. 15. Any equilateral figure inscribed in a circle must be equiangular, because the angles are contained in equal arcs, and therefore stand on equal arcs.

From the Fourth Book it follows that the circumference of a circle may be divided into 3, 4, 5, 15 parts. And by bisecting any angle standing on an arc, we can bisect the arc; consequently, we can subdivide any of the original divisions of the circle by 2, or any power of 2, as 4, 6, 8, &c. And, therefore, a polygon of a number of sides corresponding to the number of divisions in the circumference, may be inscribed in a circle, or described about it. We are not able, however, to divide geometrically a circle into seven equal parts.

It has been proved by M. Gauss, in his '*Disquisitiones Arithmeticae*,' that we are able to describe geometrically a regular polygon of  $2^n + 1$  sides, if  $2^n + 1$  be a prime number.

## A CLASSIFICATION OF THE PROPOSITIONS OF BOOK IV.

This book consists entirely of problems, the object of which is to divide the circumference of the circle into a certain number of equal parts. When, for example, we inscribe in a circle an equilateral triangle, a square, a regular pentagon, &c, we in effect divide the circumference into 3, 4, 5, &c. equal parts. 'The general design,' says De Morgan, 'is somewhat concealed by the first propositions relating to any triangle: it so happens that it is as easy to divide the circle into three parts, which are in the ratio of the angles of a triangle, as into three equal parts.' We shall classify, then, the propositions of this book, according as they refer to triangles, squares, pentagons, &c.



**Problems.***A. Relating to Triangles.*

1. In a given circle to inscribe a triangle equiangular to a given triangle . . . . . **IV. 2.**
2. About a given circle to describe a given triangle equiangular to a given triangle . . . . . **IV. 3.**
3. To inscribe a circle in a given triangle . . . . . **IV. 4.**
4. To describe a circle about a given triangle . . . . . **IV. 5.**

*B. Relating to Squares.*

1. To inscribe a square in a given circle . . . . . **IV. 6.**
2. To describe a square about a given circle . . . . . **IV. 7.**
3. To inscribe a circle in a given square . . . . . **IV. 8.**
4. To describe a circle about a given square . . . . . **IV. 9.**

*C. Relating to Pentagons.*

1. To inscribe an equilateral and equiangular pentagon in a given circle . . . . . **IV. 11.**
2. To describe an equilateral and equiangular pentagon about a given circle . . . . . **IV. 12.**
3. To inscribe a circle in a given equilateral and equiangular pentagon . . . . . **IV. 13.**
4. To describe a circle about a given equilateral and equiangular pentagon . . . . . **IV. 14.**

*D. Relating to a Hexagon.*

1. To inscribe an equilateral and equiangular hexagon in a given circle . . . . . **IV. 15.**

*E. Relating to a Quindecagon.*

1. To inscribe an equilateral and equiangular quindecagon in a given circle . . . . . **IV. 16.**

Propositions 1 and 10 do not fall in with any of these divisions, being simply employed in preparing the way for the following propositions.

**ON THE FIFTH BOOK.**

This book, which introduces a new criterion of magnitude, differs entirely from the four preceding books. In them we had to consider whether one magnitude was greater, less than, or equal to, another; but in this we consider *how many times* one is greater than the other. In other words, we here treat of *relative*, and not of *absolute* magnitude.

Although the definitions and propositions of Book v. are applied by Euclid only to geometry, and are restricted to lines, figures, &c., they are of general application, and are equally true of *number*, or, in short, of any species of quantity. These demonstrations are capable, therefore, of being treated algebraically, like those of Book II.

**Def. 1.** The word 'part,' in this definition, signifies not *any* part, but an

*aliquot* part, or *submultiple*. Thus, a line of five inches is a part, in this acceptance of the term, of a line of fifteen inches; but it is not a part of a line of fourteen inches, though in the ordinary meaning of the term it would be.

The magnitudes spoken of in the first two definitions must, of course, always be of the *same kind*. We cannot compare a line with a surface, or a surface with a solid. A line can only be compared with a line, a surface with surface, and a solid with a solid.

**Def. 3.** 'Euclid defines Ratio to be the relation of two magnitudes of the same kind one to another in respect of quantity, and he defines Proportion to be the sameness or similitude of ratios. (Def. 8.) Yet this is not enough, for he has recourse to a definition to explain what is meant by saying that one magnitude has a ratio to another, and again he finds himself obliged in like manner to expound at length the meaning of the expression "sameness of ratio." Now it is evident that if the definition of ratio were in the first instance adequate there would be no necessity for an explanation of the words "same ratio." . . . It is probable that Euclid was betrayed into vagueness of speech on this subject by his desire to avoid any reference to the idea of number, which, in rendering his explanation of ratio perfectly clear, would at the same time have restricted it to commensurable quantities, or those capable of numerical comparison. In adopting a more general mode of expression, applicable alike to commensurable and incommensurable quantities, he has not only become obscure, but, what is worse, he has offered a definition of sameness of ratio or proportion which is not an elemental or self-evident truth, but a theorem requiring proof and elucidation.'—*Cooley*.

The objections here referred to have led many commentators to reject Defs. 3 and 8 as useless. Simson considers that they are 'not Euclid's, but added by some unskilful editor;' and Dr. Barrow, in his *Lectures*, describes them as metaphysical and not properly mathematical definitions, as nothing in mathematics depends upon or can be deduced from them.' (Lect. III.)

**Def. 5.** This definition is intended to furnish a criterion by which we may determine when two ratios are equal. For the better understanding of this definition, which is the keystone of Euclid's doctrine of ratio, the student is referred to the corresponding algebraical statement given in the Appendix. It would be also well if he would read the first and last propositions of the Sixth Book, which may be considered as a commentary on this definition.

The accuracy of this definition consists in its being applicable to *incommensurable* as well as *commensurable* magnitudes. By commensurable magnitudes are meant those which are capable of being measured by the same magnitude, or which have an exact common measure; and those magnitudes are incommensurable which have no common measure.

**Def. 11.** This definition, usually called *the definition of compound ratio*, was supplied by Simson; for though Euclid used the term, he did so without defining it.

'No definition of compound ratio is to be found in any copy of the Greek text of Euclid, though it is clear that a technical use of the term was intended in the enunciation of Prop. 23, Book VI.: "Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides." It may easily be seen that the ratio of two equiangular parallelograms is in some way dependent on the ratios of their sides; but without a definition it would be impossible to infer from the above enunciation what pre-

cise relation between them it was proposed to establish. To supply this defect, Dr. Simson framed a definition, founded upon the demonstrated result from which the truth of the enunciation is inferred in Euclid's proposition.—*Ward's Fifth Book of Euclid proved Algebraically.*

Def. 12—17. The substance of these definitions may be familiarly illustrated by means of numbers. Thus, if we consider the magnitudes, A, B, C, D, to be represented respectively by the numbers 15, 6, 10, and 4, and

|                    |                          |       |                                |
|--------------------|--------------------------|-------|--------------------------------|
|                    | If $A : B :: C : D$      | or if | $15 : 6 :: 10 : 4$ ; then      |
| <i>Alternando</i>  | $A : C :: B : D$         | and   | $15 : 10 :: 6 : 4$             |
| <i>Invertendo</i>  | $B : A :: D : C$         | and   | $6 : 15 :: 4 : 10$             |
| <i>Componendo</i>  | $A + B : B :: C + D : D$ | and   | $15 + 6 : 6 :: 10 + 4 : 4$ }   |
|                    |                          |       | i.e. $21 : 6 :: 14 : 4$ }      |
| <i>Dividendo</i>   | $A - B : B :: C - D : D$ | and   | $15 - 6 : 6 :: 10 - 4 : 4$ }   |
|                    |                          |       | i.e. $9 : 6 :: 6 : 4$ }        |
| <i>Convertendo</i> | $A : A - B :: C : C - D$ | and   | $15 : 15 - 6 :: 10 : 10 - 4$ } |
|                    |                          |       | i.e. $15 : 9 :: 10 : 6$ }      |

Def. 18—20. The two kinds of proportion here referred to may be similarly illustrated thus:—

(1) In the two ranks of magnitudes—

|                                  |       |                             |
|----------------------------------|-------|-----------------------------|
| A, B, C, D;                      | or    | 6, 9, 15, 12;               |
| P, Q, R, S.                      |       | 10, 15, 25, 20;             |
| If $A : B :: P : Q$ ;            | or if | $6 : 9 :: 10 : 15$ ;        |
| $B : C :: Q : R$ ; and           |       | $9 : 15 :: 15 : 25$ ;       |
| $C : D :: R : S$                 |       | $15 : 12 :: 25 : 20$ ; then |
| <i>Ex aequo</i> $A : D :: P : S$ |       | $6 : 12 :: 10 : 20$ .       |

(2) In the two ranks of magnitudes—

|                                                                       |       |                             |
|-----------------------------------------------------------------------|-------|-----------------------------|
| A, B, C, D, E;                                                        | or    | 24, 36, 18, 16, 60;         |
| P, Q, R, S, T.                                                        |       | 12, 45, 40, 20, 80.         |
| If $A : B :: S : T$ ;                                                 | or if | $24 : 36 :: 20 : 30$ ;      |
| $B : C :: R : S$ ;                                                    |       | $36 : 18 :: 40 : 20$        |
| $C : D :: Q : R$ ;                                                    |       | $18 : 16 :: 45 : 40$ ;      |
| $D : E :: P : Q$ .                                                    |       | $16 : 60 :: 12 : 45$ ; then |
| <i>Ex aequo, in pro-<br/>portione perturbata</i> } $A : E :: P : T$ . |       | $24 : 60 :: 12 : 30$ .      |

The axioms in this book are additions by Simson. They are introduced into five of the propositions, and are useful in adding to the precision of the demonstrations.

V. 1. This proposition has already been assumed in III. 20, case i., where the sum of the doubles of two angles was taken for granted to be double of their sum; and in II. 8, where the sum of the quadruples of two figures is taken for granted to be quadruple of their sum.

V. 1, 2, 3, 5, 6. 'These are simple propositions of concrete arithmetic, covered in language which makes them unintelligible to modern ears. The first, for instance, states no more than that *ten acres and ten roods make ten times as much as one acre and one rood.*'—*De Morgan.*

V. 5. This proposition has been assumed in III. 20, case iii., where the difference of the doubles of two angles was taken for granted to be double of their difference.

**V. D.** This proposition may be termed a conditional converse of the preceding. The exact converse would be to the second as the third to the fourth; the first shall be the same multiple of the second, or the same part of it, that the third is of the fourth. But that would not be always true; for the first need not be a multiple or part of the second, nor the third of the fourth.

**V. 10.** 'This proposition is to ratios what Axiom 1, Book I. is to magnitudes.'—*Lardner*.

The propositions A, B, C, D, E were added by Simson, and have been continued in most editions of the Elements since his time.

#### A CLASSIFICATION OF THE PROPOSITIONS OF BOOK V.

'The propositions of the fifth book might be divided into four sections. Propositions 1 to 6 relate to the properties of equimultiples. Propositions 7 to 10 and 13 and 14 connect the notion of *ratio* of magnitudes with the ordinary notions of greater, equal, and less. Propositions 11, 12, 15, and 16 may be considered as introduced to show that, *if four quantities of the same kind be proportionals, they will also be proportionals when taken alternately*. The remaining propositions show that magnitudes are proportionals by *composition*, by *division*, and *ex æquo*.

'In this division of the Fifth Book, Propositions 13 and 15 are supposed to be placed immediately after Prob. 10.; and they might be taken in this order without any change in Euclid's demonstration.'—*Todhunter*.

This division will give rise to the following classification:—

#### Theorems.

##### A. Relating to the Multiples of Magnitudes.

1. If any number of magnitudes be equimultiples of as many, each of each; what multiple soever any one of the first is of its part, the same multiple shall all the first magnitudes taken together, be of all the others taken together . . . . . **V. 1.**

2. If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder is the same multiple of the remainder, that the whole is of the whole . . . . . **V. 5.**

3. If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two; the remainders are either equal to these others, or equimultiples of them . . . . . **V. 6.**

4. If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth . . . . . **V. 2.**

5. If the first be the same multiple of the second which the third is of the fourth; and if of the first and third there be taken equimultiples; these shall be equimultiples, the one of the second, and the other of the fourth . . . . . **V. 3.**

6. If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then any equimultiples whatever

of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz. the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth . . . . . V. 4.

*B. Relating to Ratios of Equality or Inequality.*

1. If four magnitudes are proportionals; if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less . . . . . V. A.

2. If the first be the same multiple or submultiple of the second that the third is of the fourth; the first is to the second as the third is to the fourth . . . . . V. C.

3. If the first be to the second as the third is to the fourth, and if the first be a multiple or submultiple of the second; the third is the same multiple or submultiple of the fourth . . . . . V. D.

4. If four magnitudes are proportionals, they are proportionals when taken separately. . . . . V. B.

5. Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes . . . . . V. 7.

6. Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio, are equal to one another . . . . . V. 9.

7. Of two unequal magnitudes the greater has a greater ratio to any other magnitude than the less has; and the same magnitude has a greater ratio to the less of two other magnitudes than it has to the greater . . . . . V. 8.

8. That magnitude which has a greater ratio than another has to the same magnitude, is the greater of the two; and that magnitude to which the same has a greater ratio than it has to another magnitude, is the lesser of the two . . . . . V. 10.

9. If the first have to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first also has to the second a greater ratio than the fifth has to the sixth . . . . . V. 13.

10. If the first have the same ratio to the second which the third has to the fourth; then if the first be greater than the third, the second is greater than the fourth; and if equal, equal; and if less, less . . . . . V. 14.

*C. Relating to Alternate Ratios.*

1. Ratios that are equal to the same ratio are equal to one another. . . . . V. 11.

2. If any number of magnitudes of the same kind be proportionals, as one of the antecedents is to its consequent, so are all the antecedents taken together to all the consequents . . . . . V. 12.

3. Magnitudes have the same ratio to one another which their equimultiples have . . . . . V. 15.

4. If four magnitudes of the same kind be proportionals, they are also proportionals when taken alternately . . . . . V. 16.

*D. Of Proportionals by Composition, Division, and Ex Aequo.*

1. If magnitudes, taken jointly, be proportionals, they are also proportionals when taken separately . . . . . V. 17.
2. If magnitudes, taken separately, be proportionals, they are also proportionals when taken jointly . . . . . V. 18.
3. If a whole be to a whole as a magnitude taken from the first is to a magnitude taken from the other; the remainder is to the remainder as the whole to the whole . . . . . V. 19.
4. If four magnitudes be proportionals, they are also proportionals by conversion . . . . . V. E.
5. If there be three magnitudes, and other three, which, taken two and two, have the same ratio; then if the first be greater than the third, the fourth is greater than the sixth; and if equal, equal; and if less, less . . . . . V. 20.
6. If there be three magnitudes, and other three which have the same ratio taken two and two, but in a cross order; then if the first magnitude be greater than the third, the fourth is greater than the sixth; and if equal, equal; and if less, less . . . . . V. 21.
7. 'Ex æquali,' or 'Ex æquo' . . . . . V. 22.
8. 'Ex æquali in proportione perturbatâ,' or, 'Ex æquo perturbato' . . . . . V. 23.
9. If the first have to the second the same ratio which the third has to the fourth; and the fifth to the second, the same ratio which the sixth has to the fourth; the first and fifth together have to the second, the same ratio which the third and sixth together have to the fourth . . . . . V. 24.
10. If four magnitudes of the same kind be proportionals, the greatest and least of them together are greater than the other two together . . . . . V. 25.
11. Ratios which are compounded of the same ratios are equal to one another . . . . . V. F.
12. If several ratios be equal to several other ratios, each to each, the ratio which is compounded of ratios which are equal to the first ratios, each to each, is equal to the ratio compounded of ratios which are equal to other ratios, each to each. . . . . V. G.
13. If a ratio which is compounded of several ratios be equal to a ratio which is compounded of several other ratios, &c. . . . . V. H.
14. If there be any number of ratios, and any number of other ratios such that the ratio which is compounded of ratios which are equal to the first ratios, each to each, &c. . . . . V. K.

## ON THE SIXTH BOOK.

This is a valuable book, as it exemplifies the application of the doctrine of ratio and proportion—the general principles of which formed the subject of the preceding book—to the comparison of the sides of similar triangles and the areas of plane rectilineal figures.

**Def. 1.** This definition, so far as it refers to triangles, is redundant; for if triangles be equiangular, it follows as a consequence that the sides about the

equal angles must be proportional. Hence only one condition is necessary to similar triangles, viz. that they be equiangular.

Def. 3. Lardner substitutes the following as preferable:—‘A straight line is said to be cut in extreme and mean ratio, when the whole line is to one segment as that segment is to the remaining one.’

Def. 4. This definition appears to be too generally expressed, as in the first six books altitude is confined to triangles and parallelograms only.

VL 1. ‘The rationale of this proposition is best understood, after comparison with the last of the book, by the following enunciation—“If there be two species of magnitudes, named A and B, and if every A be accompanied by one B, and one only; and if every B be accompanied by one A, and one only; and if, according as one A is greater than, equal to, or less than another, the B of the first is greater than, equal to, or less than that of the second; and if, when one A is a multiple of another, the B of the first is the same multiple of the second; then any two A’s have always the same ratio to one another as their corresponding B’s.”’—*De Morgan*.

VL 2. ‘The enunciation of this proposition is inaccurate in several respects. In the first part, the manner in which the parallel cuts the sides is not distinctly described, and the second part is, strictly speaking, *false*, inasmuch as a line may cut two sides *proportionally*, and yet not be parallel to the third side.’—*Lardner*.

The same commentator suggests that the enunciation should be changed in the following manner:—1. *If a line be drawn parallel to any side of a triangle, it divides the other sides, or those produced so that their segments between the parallel and the third side shall have the same ratio to their segments between the parallel and the vertex of the opposite angle; and 2. If a line cut the two sides in this manner, it will be parallel to the third side.*

VL A. This is Simson’s. It consists of two theorems, the converse of each other. It is not, however, universally true; for example, in an equilateral triangle the line which bisects the external angle is parallel to the base, and can never meet it. The property specified, indeed, is only true of scalene triangles that are not right-angled.

VL 4. This proposition is the first case of similar triangles. It is a generalisation of the third case of equal triangles (Prop. I. 26), which refers to *equal* triangles, while this proposition refers to *similar* triangles. By ‘homologous’ sides are meant those which are the antecedents or the consequents of the ratios.

VL 5. By Def. 1 two things are necessary to similar figures, viz. that the angles be equal, and the sides proportional. As this definition applies to triangles, it is, as has been before remarked, tautological, for by Prop. 4 it is proved that if the angles be equal, the sides are proportional; and conversely, by Prop. 5, if the sides be proportional, the angles are equal.

This proposition corresponds to I. 8 in the same way that the preceding one does to I. 26.

VL 4 and 5 and I. 47 and 48 are the most important propositions in the Elements, since it is by them we are enabled to apply algebra to geometry, and also to solve all problems relating to rectilineal figures, inasmuch as all such figures may be resolved into triangles, and triangles may be resolved

into two right-angled triangles by drawing a perpendicular from the vertex to the base.

VL. 6 corresponds to L. 4, and is the third case of similar triangles.

VL. 9. The word 'part' here is used in the sense in which it is employed in v. def. 1. L. 10 is a particular case of this problem.

VL. 10. To divide a line similarly to another is to divide it so that its segments are proportional to those of the other. The preceding proposition is a particular case of this.

VL. 11. The student will easily perceive that the method here given of finding a third proportional to two given straight lines, may be extended to a series of ratios in progression. A repetition of the same construction as here given will solve the problem.

This is a particular case of the next proposition.

VL. 13. This proposition enables us to find 3, 7, 15, &c., means between two given straight lines. For we may insert one between the first and second, and one between the second and third. And thus we may continue to find means between the terms to any extent.

We are not able, by geometry, to find two mean proportionals between two given lines, though various practical contrivances have been suggested to solve this problem, which was much discussed by the ancients. One of the earliest of these was by Plato, another by Philo, and another, in more modern times, by Descartes.

VL. 14. 'Owing to the disjointed manner in which Euclid treats compound ratio, this proposition is strangely out of place. It is a particular case of vl. 23, being that in which the ratio of the sides, compounded, gives a ratio of equality. The proper definition of four magnitudes being reciprocally proportional, is that the ratio compounded of their ratios is that of equality.'—*De Morgan*.

VL. 16. This is an important proposition, as many useful results grow out of it. To mention only one, it is the foundation of the operation in arithmetic known as the Rule of Three, that is, in which three terms, being given, it is required to find a fourth proportional.

This proposition is a case of vl. 14.

VL. 18. We practically make use of this proposition in drawing plans, maps, &c. The converse of this is in vl. 20.

VL. 19. By 'duplicate ratio' is meant the ratio of their squares, or that ratio which arises from the composition of two equal ratios. The enunciation might be thus expressed: Similar triangles are to one another as the squares of their corresponding or like sides. From vl. 1 it follows that triangles are in the ratio compounded of their bases and altitudes. When they are similar, this compound ratio becomes the duplicate ratio.

VL. 20. This proposition furnishes the method of finding the area of a regular polygon and a circle.

VL. 22. 'This proposition is equivalent to stating that if two ratios be equal, their duplicates and subduplicates will also be equal.'—*Lardner*.

VL. 23. 'The use of compound ratio consists wholly in this, that by means of it circumlocutions may be avoided, and thereby propositions may be more briefly either enunciated or demonstrated, or both may be done; for instance, if this 23rd proposition of the Sixth Book were to be enunciated, without mentioning compound ratio, it might be done as follows: *If two parallel*



ograms be equiangular, and if as a side of the first to a side of the second, so any assumed straight line be made to a second straight line; and as the other side of the first to the other side of the second, so the second straight line be made to a third. The first parallelogram is to the second, as the first straight line to the third.'—*Simson*.

VI. 25. This is a very useful problem, as by it we can construct a figure of a different shape to another, while its quantity is the same; as for example an equilateral triangle may be made equal to a square.

VI. 27, 28, 29. These propositions are not considered of much importance by many geometers. Dechales, Lardner, and others, recommend that they should be omitted. *Simson*, on the other hand, thinks them 'the most general and useful of all in the Elements,' and asserts that they are 'most frequently made use of by the ancient geometers in the solution of other problems.'

VI. 31. This is the general proposition, of which I. 47 is only a particular case.

VI. 33. 'The second part of the proposition is an addition of Theon's, as he tells us in his commentary on Ptolemy's *Μεγάλῃ Συναρίσι*, p. 50.'—*Simson*.

VI. B, C, D, were added by *Simson*.

## A CLASSIFICATION OF THE PROPOSITIONS OF BOOK VI.

### Problems.

1. From a given straight line to cut off any part required . . . . . VI. 9.
2. To divide a given straight line similarly to a given divided straight line, that is, into two parts that shall have the same ratios to one another which the parts of the divided given straight line have . . . . . VI. 10.
3. To find a third proportional to two given straight lines . . . . . VI. 11.
4. To find a fourth proportional to three given straight lines . . . . . VI. 12.
5. To find a mean proportional between two given straight lines . . . . . VI. 13.
6. To cut a given straight line in extreme and mean ratio . . . . . VI. 30.
7. To a given straight line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram similar to a given parallelogram; but the given rectilineal figure to which the parallelogram to be applied is to be equal, must not be greater than the parallelogram applied to half of the given line, having its defect similar to the defect of that which is to be applied; that is, to the given parallelogram . . . . . VI. 28.
8. To a given straight line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogram similar to another given . . . . . VI. 29.
9. Upon a given straight line to describe a rectilineal figure similar, and similarly situated to a given rectilineal figure . . . . . VI. 18.
10. To describe a rectilineal figure which shall be similar to one, and equal to another given rectilineal figure. . . . .

### Theorems.

#### A. Relating to Similar Triangles.

1. Similar triangles are to each other in the duplicate ratio of their homologous sides. . . . . VI. 19.

2. In equiangular triangles the sides about the equal angles are proportional, and the sides which are opposite to the equal angles are homologous . . . . . VI. 4.
3. Conversely, if two triangles have their sides proportional they are equiangular, and the equal angles are subtended by the homologous sides . . . . . VI. 5.
4. If two triangles have one angle in each equal, and the sides about the equal angles proportional, the triangles are equiangular, and have those angles equal which the homologous sides subtend . . . . . VI. 6.
5. If two triangles have one angle in each equal, the sides about two other angles proportional, and each of the remaining angles either less or not less than a right angle, the triangles are equiangular, and those angles are equal about which the sides are proportional . . . . . VI. 7.
6. In a right-angled triangle, if a perpendicular be drawn from the right angle upon the opposite side, it divides the triangle into parts which are similar to the whole and to one another . . . . . VI. 8.
7. Equal triangles, which have one angle in each equal, have the sides about the equal angles reciprocally proportional; . . . . . VI. 15.
- Conversely, two triangles which have an angle equal, and the sides about the equal angles reciprocally proportional, are equal . . . . .
8. If two triangles have two sides proportional, and be so placed at an angle that the homologous sides are parallel, the remaining sides form one straight line . . . . . VI. 32.

*B. Relating to Similar Parallelograms.*

1. Equiangular parallelograms are to each other in the ratio compounded of the ratio of their sides . . . . . VI. 23.
2. In any parallelogram the parallelograms which are about the diagonal are similar to the whole and to each other . . . . . VI. 24.
3. If two similar parallelograms have a common angle, and be similarly situated; they are about the same diameter . . . . . VI. 26.
4. Equal parallelograms which have one angle in each equal, have the sides about the equal angles reciprocally proportional; . . . . . VI. 14.
- And conversely, parallelograms which have one angle in each equal, and the sides about them reciprocally proportional, are equal . . . . .

*C. Relating to Similar Polygons and Rectilineal Figures in general.*

1. Similar polygons may be divided into similar triangles equal in number and proportional to the polygons; and the polygons are to each other in the duplicate ratio of their homologous sides . . . . . VI. 20.
2. Rectilineal figures which are similar to the same figure are similar also to each other . . . . . VI. 21.
3. If four straight lines be proportional, the similar rectilineal figures similarly described on them are also proportional; . . . . . VI. 22.
- And conversely, if four similar rectilineal figures, similarly described on four straight lines, be proportional, the straight lines are also proportional . . . . .
4. If any similar rectilineal figures be similarly described on the sides of a right-angled triangle, the figure described on the side sub-

tending the right angle is equal to the sum of the figures on the other side . . . . . VI. 31.

*D. Relating to the Comparison of Parallelograms with Triangles.*

1. If triangles and parallelograms have the same altitude, they are to one another as their bases . . . . . VI. 1.

*E. Relating to the Proportions arising from Lines drawn in Triangles.*

1. If a straight line be drawn parallel to any side of the triangle, it divides the other sides, or those sides produced, proportionally; And conversely, if a straight line divide the sides of a triangle, or those sides produced, proportionally, it is parallel to the remaining side . . . . . VI. 2.

2. A straight line bisecting the angle of a triangle divides the opposite side into segments proportional to the conterminous sides; And conversely, if a straight line drawn from any angle of a triangle divide the opposite side into segments proportional to the conterminous sides, it bisects the angle . . . . . VI. 3.

*F. Relating to Rectangles formed by Straight Lines that are Proportionals.*

1. If three straight lines be proportionals, the rectangle under the extremes is equal in area to the square on the mean; And conversely, if in three straight lines the rectangle under the extremes is equal in area to the square on the mean, the lines are proportionals . . . . . VI. 17.

2. If four straight lines be proportionals, the rectangle under the extremes is equal in area to the rectangle under the means; And conversely, if in four straight lines the rectangle under the extremes be equal in area to the rectangle under the means, the lines are proportionals . . . . . VI. 16.

*G. Relating to Circles.*

1. If angles are in equal circles, whether at the centres or circumferences, they have the same ratio which the circumferences on which they stand have to one another . . . . . VI. 33.

*H. Relating to inscribed Figures.*

1. If a parallelogram be constructed on the half of one of the sides of a triangle in which it is inscribed, it is the greatest parallelogram which can be inscribed in the triangle . . . . . VI. 27.

## ON THE ELEVENTH BOOK.

In the first six books the figures are supposed to be drawn in the same plane, but in this book we consider the properties of lines which are in different planes, and also the relative position of planes.

*Def. 8.* It ought to be added, that the planes do not meet, though produced

both ways, for they might be produced in *one* direction and not meet, and yet be not parallel.

**Def. 10.** This definition does not appear to be quite correct, as there are some solid figures contained by similar plane figures in the same number and of the same magnitude that are neither similar nor equal. This defect has led some geometers to reject this definition as an interpolation.

**Def. 11.** The following improvement of this definition has been suggested: 'Of solid figures contained by *planes*, those are similar which have all their solid angles equal, each to each, and which are contained by the same number of similar plane figures, similarly situated.'—*J. Thomson, LL.D.*

**XL 3.** This proposition presents a straight line to us in a new aspect.

'When the walls, floor, and ceiling of an apartment are planes, the lines in which they cut each other are straight lines. So likewise are the ridge and the eaves of a house, when the walls and the sloping surfaces of the roof are planes.'—*James Thomson, LL.D.*

**XL 7.** Simson thinks this proposition has been put into this book by some unskilful editor.

As in Book II. a rectangle was said to be contained by two of its sides which contain one of its right angles, so a right-angled parallelopiped in the same way may be said to be contained by the three straight lines which contain any one of its solid angles. And as a numerical expression was found for a rectangle, by multiplying the length of the conterminous sides together, so a similar expression may be found for a right-angled parallelopiped, by multiplying the length of the three sides which contain any one of the solid angles. Thus, if the three adjacent edges of a rectangular parallelopiped be respectively 4, 5, and 6 feet, the product of these three numbers, viz.  $4 \times 5 \times 6 = 120$ , feet, will be the number of cubic feet contained in the solid. And if the three sides be equal to one another, that is, if the figure be a cube, the volume will be found by cubing the length of the sides.

A superficies was in Book I. defined to be extension in two directions, and similar plane rectilinear figures were shown in Book VI. to be to one another in the duplicate ratio or as the squares of their homologous sides. In a similar manner, a solid may be considered as extension in *three* directions, and similar solids may be shown to be to one another in the triplicate ratio, or as cubes of their homologous edges.

## A CLASSIFICATION OF THE PROPOSITIONS OF BOOK XI.

### Problems.

1. To draw a straight line perpendicular to a plane, from a given point above it . . . . . **XL 11.**
2. To erect a straight line at right angles to a given plane, from a point given in the plane . . . . . **XL 12.**

### Theorems.

#### A. Relating to the Position of Straight Lines with respect to Planes.

1. One part of a straight line cannot be in a plane, and another part above it . . . . . **XL 1.**

2. If two straight lines cut one another, they are in one plane; and three straight lines which meet are in one plane, or in other words, only one plane can be drawn through three points which are not on the same right line \* . . . . . **XI. 2.**
3. If a straight line stand at right angles to each of two straight lines in the point of their intersection, it is also at right angles to the plane through them, that is, to the plane in which they are . . . . . **XI. 4.**
4. If three straight lines meet all in one point, and a straight line stand at right angles to each of them in that point, these three straight lines are in one and the same plane . . . . . **XI. 5.**
5. If two straight lines be at right angles to the same plane, they are parallel to one another . . . . . **XI. 6.**
6. If two straight lines be parallel, the straight line drawn from any point in the one to any point in the other, is in the same plane with the parallels . . . . . **XI. 7.**
7. If two straight lines be parallel, and one of them be at right angles to a plane; the other also shall be at right angles to the same plane . . . . . **XI. 8.**
8. Two straight lines which are each of them parallel to the same straight line, and not in the same plane with it, are parallel to one another . . . . . **XI. 9.**
9. If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the first two; the first two and the other two shall contain equal angles . . . . . **XI. 10.**
10. If two straight lines be drawn from a given point either in or above a plane, they cannot both be at right angles to it . . . . . **XI. 13.**

*B. Relating to the Position of Planes with respect to each other*

1. If two planes cut one another, their common section is a straight line . . . . . **XI. 3.**
2. If two planes which cut one another be each of them perpendicular to a third plane, their common section is perpendicular to the same plane . . . . . **XI. 19.**
3. Planes to which the same straight line is perpendicular, are parallel to one another . . . . . **XI. 14.**
4. If two straight lines meeting one another be parallel to two other straight lines which meet one another, but are not in the same plane with the first two; the plane which passes through these is parallel to the plane passing through the others . . . . . **XI. 15.**
5. If two parallel planes be cut by another plane, their common sections with it are parallels . . . . . **XI. 16.**
6. If two straight lines be cut by parallel planes, they shall be cut in the same ratio . . . . . **XI. 17.**
7. If a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane . . . . . **XI. 18.**

\* Three points not in the same straight line are therefore sufficient to determine a plane.

*C. Relating to Solid Angles.*

1. If a solid angle be contained by three plane angles, any two of them are greater than the third . . . . . **XI. 20.**
2. Every solid angle is contained by plane angles, which together are less than four right angles . . . . . **XI. 21.**

**ON THE TWELFTH BOOK.**

In Book XII., of which only two propositions are here given, introduces a new mode of comparing magnitudes. It is that called 'the method of exhaustion,' which was often employed by the ancient geometers, but which is now discarded as too tedious. XII. 2 is, however, interesting as an example of this method.

## GEOMETRICAL ANALYSIS.

The truths of geometry were gradually made known to mankind by patient and laborious investigation. The means or instrument by which the ancients obtained these results is called *Geometrical Analysis*, because it was conducted purely on geometrical principles, and without the aid of any of the other branches of the mathematics which are now in use among the moderna. The ancient analysis, though not so expeditious as the modern, which has the assistance of algebra, is nevertheless worthy of attention on account of its purity and rigour.

Analysis may be defined to be the separation or *resolution* of a whole into its component parts. Synthesis, which is the reverse of this, may be said to be the *composition* or making up a whole out of its parts. This latter is the method chiefly used in the Elements.

In the analysis of a *problem*, we commence by assuming the *quæsitæ* or what is required to be done, and by a process of reasoning we arrive at the data or some conclusion which may furnish us with the means of solving the problem. In the synthesis, we begin where we left off in the analysis, and retrace our steps until we come to where we started at first, viz. the solution of the problem.

In analysing a *theorem*, our object is to ascertain whether the statement be true or not. We therefore in the first place, assume it to be true; and by a chain of deductions we arrive at some result which is either an admitted truth or an evident fallacy. If the final conclusion be some demonstrated truth, the statement is true, and by retracing our steps we may prove the proposed theorem. If, however, the final result contradict some established truth, then the statement is false.

Previous to the pupil attempting the analysis of new propositions, he should accustom himself to retrace the steps of some of the propositions of Euclid from the end to the beginning, endeavouring to find some clue which might suggest the mode of demonstration. This will serve as an introduction to the analytical method. Thus, take for example i. 32, in which we are required to prove that the exterior angle is equal to the two interior and opposite angles. First, draw the line CE, assuming that it divides the exterior angle into parts that are respectively equal to the two interior and opposite angles. It becomes at once evident that the line thus drawn must be parallel to BA, the side of the triangle, which is the necessary construction to make in order to prove the theorem.

No very useful directions can be given for conducting an analysis, each instance mostly requiring a treatment peculiar to itself. The student must rely chiefly on his own penetration to discover the steps to be taken; and it will often be not until after repeated failures that the problem is finally solved. Nor should these unsuccessful attempts be thought altogether useless; such attempts have sometimes led to the discovery of other propositions.

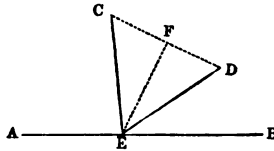
An example will perhaps better illustrate the manner of conducting an analysis.

PROP. I. PROBLEM.

*To find a point in a given straight line, which shall be at an equal distance from two other given points.*

Let AB be the given straight line, and C and D the given points.

*Analysis.*—Assume that the point E is equally distant from the points C and D.



Join CD, CE, DE, then CED is an isosceles triangle; bisect CD in F, and join FE.

Then because CE is equal to DE, and CF equal to FD, and EF common, therefore the triangle CEF is equal to the triangle DEF; (1. 8.) therefore the angle CFE is equal to the angle DFE, which are therefore right angles.

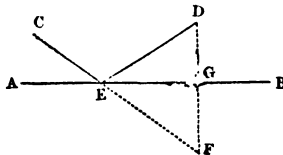
*Synthesis.*—Join the points C and D, and bisect CD in F, from the point F draw a perpendicular to the line AB in E; the point E is equally distant from C and D. Because CF is equal to FD and FE common, and the angle CFE equal to the angle DFE, therefore CE is equal to ED.

PROP. II. PROBLEM.

*From two given points on the same side of a given straight line it is required to draw two straight lines to the same point in the given straight line, which shall make equal angles with it.*

Let C and D be the given points, and AB the given straight line.

*Analysis.*—Assume CE and ED to be the two straight lines, which make equal angles with AB.



Produce CE, make EF equal to ED, and join DF.



Then because the angle DEB is equal to the angle CEA,  
and the angle CEA is equal to the angle GEF, (I. 15)  
*therefore the angle DEB is equal to the angle GEF;*  
and because the side ED is equal to the side EF (constr.) and EG is common;  
and the angle DEG is equal to the angle FEG;  
*therefore the side DG is equal to the side GF, and the angle DGE is  
equal to the angle EGF, which are consequently right angles.*

*Synthesis.*—From D draw DG at right angles to AB,  
produce DG to F, and make GF equal to DG;  
join CF, cutting AB in E; and join ED.

Then because DG is equal to GF, and EG common,  
and the angle DGE is equal to the angle FGE (constr.);  
*therefore the angle DEG is equal to the angle FEG;*  
but the angle FEG is equal to the angle CEA,  
*therefore the angle DEG is equal to the angle CEA.*

# QUESTIONS AND EXERCISES

ON

## BOOK I.

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### DEFINITIONS.

I. Define a point. II. Define a line. III. What are the extremities of a line? IV. What is a straight line? V. Define a superficies. VI. The extremities of a superficies. VII. A plane superficies. VIII. A plane angle. IX. A plane rectilineal angle. X. A right angle, and a perpendicular. XI. An obtuse angle. XII. An acute angle. XIII. What is a term, or boundary? XIV. Define a figure. XV. A circle. XVI. What is the centre of a circle? XVII. What is the diameter of a circle? XVIII. What is a semicircle? XIX. What is the centre of a semicircle? XX. Define a rectilineal figure. XXI. A trilateral figure or triangle. XXII. A quadrilateral figure. XXIII. A multilateral figure, or polygon. XXIV. An equilateral triangle. XXV. An isosceles triangle. XXVI. A scalene triangle. XXVII. A right-angled triangle. XXVIII. An obtuse-angled triangle. XXIX. An acute-angled triangle. XXX. A square. XXXI. An oblong. XXXII. A rhombus. XXXIII. A rhomboid. XXXIV. A trapezium. XXXV. What are parallel straight lines?

### POSTULATES.

- I. Enunciate the first postulate.
- II. Enunciate the second postulate.
- III. Enunciate the third postulate.

### AXIOMS.

- I. How are things which are equal to the same thing related to one another?
- II. If equals be added to equals, how are the wholes related to one another?
- III. If equals be taken from equals, how are the remainders related to one another?
- IV. If equals be added to unequals, how are the wholes related to one another?
- V. If equals be taken from unequals, how are the remainders related to one another?

- VI. How are the doubles of the same thing related to one another?
- VII. How are the halves of the same thing related to one another?
- VIII. When are magnitudes equals?
- IX. How is the whole related to its part?
- X. How are right angles related to one another?
- XI. Under what circumstances will two straight lines meet, if they are continually produced, and on what side will they meet?

## ENUNCIATIONS AND COROLLARIES OF THE PROPOSITIONS OF BOOK I.

### FOR EXAMINATION APART FROM THE TEXT.

Prop. 1. To describe an equilateral triangle upon a given finite straight line.

Prop. 2. From a given point to draw a straight line equal to a given straight line.

Prop. 3. From the greater of two given straight lines to cut off a part equal to the less.

Prop. 4. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, they shall likewise have their bases, or *third sides*, equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, viz., those to which the equal sides are opposite.

Prop. 5. The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall be equal.

Cor. Every equilateral triangle is also equiangular.

Prop. 6. If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to the equal angles, are equal to one another.

Cor. Every equiangular triangle is also equilateral.

Prop. 7. Upon the same base, and on the same side of it, there cannot be two triangles having their sides terminated in one extremity of the base, equal to one another, and likewise those which are terminated in the other extremity.

Prop. 8. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them of the other.

Prop. 9. To bisect a given rectilineal angle; that is, to divide it into two equal angles.

Prop. 10. To bisect a given finite straight line; that is, to divide it into two equal parts.

Prop. 11. To draw a straight line at right angles to a given straight line, from a given point in the same.

Prop. 12. To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Prop. 13. The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Prop. 14. If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

Prop. 15. If two straight lines cut one another, the vertical or opposite angles shall be equal.

Cor. 1. If two straight lines cut one another, the angles they make at the point where they cut, are together equal to four right angles.

Cor. 2. All the angles made by any number of lines meeting in one point are together equal to four right angles.

Prop. 16. If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Prop. 17. Any two angles of a triangle are together less than two right angles.

Prop. 18. The greater side of every triangle is opposite to the greater angle.

Prop. 19. The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Prop. 20. Any two sides of a triangle are together greater than the third side.

Prop. 21. If from the ends of a side of a triangle, there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Prop. 22. To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

Prop. 23. At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Prop. 24. If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle shall be greater than the base of the other.

Prop. 25. If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides equal to them of the other.

Prop. 26. If two triangles have two angles of the one equal to two angles of the other, each to each; and one side equal to one side, viz., either the sides adjacent to the equal angles, or the sides opposite to equal angles in each; then the other sides shall be equal, each to each, and also the third angle of the one to the third angle of the other.

Prop. 27. If a straight line falling upon two other straight lines, make the alternate angles equal to one another; these two straight lines shall be parallel.

Prop. 28. If a straight line falling upon two other straight lines, make the exterior angle equal to the interior and opposite angle upon the same side of the line; or make the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.

Prop. 29. If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite angle upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

Prop. 30. Straight lines which are parallel to the same straight line, are parallel to each other.

Prop. 31. To draw a straight line through a given point parallel to a given straight line.

Prop. 32. If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

Cor. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Cor. 2. All the exterior angles of any rectilineal figure (made by producing the sides successively in the same direction), are together equal to four right angles.

Prop. 33. The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Prop. 34. The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects it, that is, divides it into two equal parts.

Prop. 35. Parallelograms upon the same base, and between the same parallels, are equal to one another.

Prop. 36. Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Prop. 37. Triangles upon the same base, and between the same parallels, are equal to one another.

Prop. 38. Triangles upon equal bases, and between the same parallels, are equal to one another.

Prop. 39. Equal triangles upon the same base, and upon the same side of it, are between the same parallels.

Prop. 40. Equal triangles upon equal bases in the same straight line, and towards the same parts, are between the same parallels.

Prop. 41. If a parallelogram and a triangle be upon the same base, and between the same parallels; the parallelogram shall be double of the triangle.

Prop. 42. To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Prop. 43. The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.

Prop. 44. To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Prop. 45. To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Cor. To a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure.

Prop. 46. To describe a square upon a given straight line.

Cor. Every parallelogram that has one right angle, has all its angles right angles.

**Prop. 47.** In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

**Prop. 48.** If the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it; the angle contained by these two sides is a right angle.

## GEOMETRICAL EXERCISES ON BOOK I.

The following Exercises, which have been selected on account of their simplicity, and their value as geometrical facts useful to the learner in more advanced investigations, have been arranged, up to Ex. 48, as far as possible in the order of the Propositions. After that number they are of a more miscellaneous character, but it is believed that none of them are beyond the capacity of the student who has gone carefully through the First Book.

1. In the figure of Euclid, Book I., Prop. 1., describe an equilateral triangle upon the other side of AB.

2. If in the same figure, AB be produced both ways to meet the circles in D and E, and from C, the lines CD and CE be drawn; show that the triangle CDE is isosceles.

3. In the same figure, if the circles intersect in F on the other side of AB, and AF, BF be drawn; prove that ACBF is a rhombus.

4. In the same figure, if the given line is produced to meet either of the circles in P; show that P and the points of intersection of the circles, are the angular points of an equilateral triangle.

5. If in the same figure, CA, CB be produced to meet the circumference in D, E, and F be the other point of intersection of the circles, show that DF, EF are in one line.

6. Upon a given straight line describe an isosceles triangle that shall have each of its sides double of the base.

7. Prove by superposition, that if two squares have one side of the one equal to one side of the other, the squares are equal in all respects.

8. In the figure of Prop. 1. 5, if FC and BG meet at H, show that AH bisects the angle BAC.

9. In the same figure, if FC and BG meet at H, show that FH and GH are equal.

10. On a given straight line to describe an isosceles triangle, of which the perpendicular height is equal to the base.

11. In an isosceles triangle, the straight line which bisects the vertical angle also bisects the base, and is perpendicular to it.

12. In an isosceles triangle, the line drawn from the vertex to the middle point of the base bisects the vertical angle, and is perpendicular to the base.

13. If, in a triangle, the perpendicular from the vertex on the base bisect the base, the triangle is isosceles.

14. The opposite angles of a rhombus are equal.

15. A given angle  $BAC$  is bisected ; if  $CA$  be produced to  $G$ , and the angle  $BAG$  bisected, the two bisecting lines are at right angles.

16. Prove the second part of Prop. I. 16, viz., that the exterior angle  $ACD$  is greater than the interior opposite angle  $ABC$ .

17. In the figure of Prop. I. 17, show that  $ABC$  and  $ACB$  are together less than two right angles, by joining  $A$  to any point in  $BC$ .

18. The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line ; and of others, that which is nearer to the perpendicular is less than the more remote ; and two, and only two, equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.

19. The difference of any two sides of a triangle is less than the third.

20. The three sides of a triangle taken together are greater than the double of any one side, but less than the double of any two sides.

21. To make a triangle equal to a given triangle.

22. The perpendiculars let fall on two sides of a triangle from any point in the straight line bisecting the angle between them, are equal to each other.

23. If two lines bisect perpendicularly two sides of a triangle, the perpendicular from their point of section upon the base, will bisect it.

24. If a straight line falling upon two other straight lines, make the two exterior angles on the same side of it equal to two right angles, these two straight lines are parallel.

25. Any straight line parallel to the base of an isosceles triangle makes equal angles with the sides.

26. If a line be perpendicular to one of two parallel lines, it will be perpendicular also to the other.

27. Lines which are perpendicular to parallel lines are also parallel.

28. The straight line which bisects the sides of a triangle is parallel to its base.

29. If a line be drawn through the middle point of one side of a triangle, parallel to the base, prove that it will bisect the other side.

30. Prove that the triangle cut off by this parallel is one-fourth part of the whole triangle.

31. Prove that the line drawn from the vertex of a triangle, bisecting the base, bisects every parallel to the base, whether above or below the vertex.

32. Each angle of an equilateral triangle is a third of two right angles, or two-thirds of one.

33. Trisect a right angle.

34. If the straight line bisecting the exterior angle of a triangle be parallel to the base, show that the triangle is isosceles.

35. If the sides of an equilateral and equiangular hexagon, or six-sided figure, be produced till they meet, the angles formed at the points of meeting are together equal to four right angles.

36. The parts of all perpendiculars to two parallel lines, intercepted between them, are equal.

37. If the opposite sides of a quadrilateral be equal, it is a parallelogram.

38. The diagonals of a parallelogram bisect each other.
39. The parallelogram, whose diameters are equal, is rectangular.
40. ABCD is a parallelogram, E is the middle point of DC ; show that the triangle AEB is half the parallelogram.
41. Prove that the four triangles into which a parallelogram is divided by its diagonals are equal in area.
42. If the opposite sides of a trapezium be parallel to one another, the straight line joining their bisections, bisects the trapezium.
43. If, from any point within a parallelogram, straight lines be drawn to the extremities of two opposite sides, the two triangles upon these sides are together equal to half of the parallelogram.
44. The parallelograms about the diagonal of a square are squares.
45. In the figure, *z.* 43, join EH, BD, and GF; prove that the three diagonals thus drawn, are parallel to each other.
46. In the figure, *z.* 47, if BG and CH be joined, these lines will be parallel.
47. In the figure, *z.* 47, if G and H be joined, show that the triangle GAH will be equal to the given triangle ABC.
48. In a rhombus, the squares of all the sides are together equal to the squares of the diagonals.
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49. A straight line is drawn bisecting a parallelogram ABCD, and meeting AD in E and BC at F; show that the triangles EBF and CED are equal.
50. Given the diagonal of a square to construct it.
51. If the sides of a quadrilateral figure be bisected, and the points of bisection joined, the included figure is a parallelogram, and equal in area to half the original figure.
52. The square on the diameter of any square is double of it.
53. The square on the base of an isosceles triangle, whose vertical angle is a right angle, is equal to four times the area of the triangle.
54. If the diagonals of a parallelogram are equal, all its angles are equal.
55. Through a given point P, to draw a straight line, which shall cut off equal parts from two straight lines AB and AC, cutting one another in A.
56. To draw a straight line from a given point to meet another straight line, which shall make with it an angle equal to a given rectilineal angle.
57. To draw a straight line through a given point such that the part of it intercepted between two given parallel straight lines may be of given length.
58. Draw a line EF parallel to the base BC of a triangle ABC, so that EF shall be equal to BE.
59. In a given square to inscribe an equilateral triangle, having one of its angular points upon one of the angular points of the square, and its two remaining angular points one in each of two adjacent sides of the square.
60. To inscribe a square in a given right-angled isosceles triangle.
61. Given the perpendicular and the side of an isosceles triangle, to construct it.



62. If the two exterior angles at the base of a triangle be bisected, and the bisecting lines produced until they intersect, the line drawn from the point to the vertical angle will bisect it.

63. ABC is a triangle, right-angled at A, and having the angle B double of the angle C; show that the side CB is double of the side AB.

64. The lines which bisect the angles of any parallelogram, form a rectangular parallelogram, whose diameters are parallel to the sides of the former.

65. In any triangle ABC, if BE, CF be perpendiculars on any line through A, and D be the bisection of BC, show that  $DE = DF$ .

66. In any right-angled triangle, the middle point of the hypotenuse (side opposite the right angle) is equally distant from the three angles.

67. On the sides AC, BC of a triangle ABC, squares ACDE, BCFH are described; show that the straight lines AF and BD are equal.

68. The square on the side subtending an acute angle of a triangle is less than the squares on the sides containing the acute angle.

69. The square on the side subtending an obtuse angle of a triangle is greater than the squares on the sides containing the obtuse angle.

70. In the figure of Prop. I. 47 prove that if perpendiculars be let fall from F and K on BC produced, the parts produced will be equal; and the perpendiculars together will be equal to BC.

71. If two circles cut each other, the line joining their points of intersection is bisected at right angles by the line joining their centres.

72. Describe a circle which shall pass through two given points, and have its centre in a given line.

73. From two given points on the same side of a given line, draw two lines which shall meet in that line and make equal angles with it.

74. Through a given point draw a line, so that the perpendiculars upon it from two other given points may be equal to each other.

75. In a given straight line, find a point equally distant from two given points; one in, and the other without, the given straight line.

76. If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet, they will contain an angle equal to an exterior angle of the triangle.

77. If in a right-angled triangle a line be drawn dividing the right angle into two parts, which shall be respectively equal to the adjacent base angles, prove that it will bisect the hypotenuse.

78. If a line be drawn from the middle of the hypotenuse of a right-angled triangle to the right angle, prove that it will be equal to half the hypotenuse.

79. Draw a line DE, parallel to the base BC of a triangle ABC, so that DE is equal to the sum of BD and CE.

80. Draw a line DE, parallel to the base BC of a triangle ABC, so that DE is equal to the difference of BD and CE.

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# QUESTIONS AND EXERCISES

ON

## BOOK II.



### DEFINITIONS.

1. Define a rectangle. By what is it contained?
2. Define a gnomon.

### PROPOSITIONS AND COROLLARIES.

Prop. 1. If there be two straight lines, one of which is divided into any number of parts; the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Prop. 2. If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square of the whole line.

Prop. 3. If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.

Prop. 4. If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Cor. The parallelograms about the diameter of a square are likewise squares.

Prop. 5. If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

Cor. The difference of the squares of two unequal lines is equal to the rectangle contained by their sum, and their difference.

Prop. 6. If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

Prop. 7. If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Prop. 8. If a straight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square of the other part, is equal to the square of the straight line, which is made up of the whole and that part.

Prop. 9. If a straight line be divided into two equal, and also into two unequal parts, the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

Prop. 10. If a straight line be bisected, and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Prop. 11. To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

Prop. 12. In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

Prop. 13. In every triangle, the square of the side subtending either of the acute angles, is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the acute angle and the perpendicular let fall upon it from the opposite angle.

Prop. 14. To describe a square that shall be equal to a given rectilinear figure.

### ALGEBRAICAL AND ARITHMETICAL PROOFS OF THE PROPOSITIONS OF BOOK II.

The propositions of this book can be demonstrated both algebraically and arithmetically. A *line* may be represented by the *number* of linear units it contains; thus, the line AB, which contains  $a$  or 6 units, may be designated as  $a$  or 6. A *rectangle*, which in geometry is said to be contained by any two of the straight lines containing one of the right angles, may be expressed by the *product* of its length and breadth; thus, in the rectangle contained by AB, AD, if AB be equal to  $a$  or 4, and AD to  $b$  or 3, the area of the rectangle may be represented by  $ab$  or  $4 \times 3$ . And therefore if the two sides of the rectangle be equal, or if  $b$  be equal to  $a$ , the figure is a *square*, which may be represented by multiplying the length of the side into itself, as  $a \times a = a^2$ , or  $4 \times 4 = 4^2$  or 16.

An algebraical and arithmetical proof of each proposition will now be given, and the learner may be exercised in solving the problems independently by means of other letters and figures.

#### PROP. I. THEOREM.

*Algeb.*—Referring to the figure in Book II., if  $A = x$ ,  $BC = a$ ,  $BD = m$ ,  $DE = n$ , and  $EC = p$ ;

then  $a = m + n + p$ ,

multiply both sides by  $x$ ,

$\therefore ax = mx + nx + px$ .

i.e.,  $A \cdot BC = A \cdot BD + A \cdot DE + A \cdot EC$ .

*Arith.*—Again, let  $x=6$ ,  $a=12$ ,  $m=5$ ,  $n=3$ , and  $p=4$ ;  
 then  $12=5+3+4$ ,  
 multiply both sides by 6,  
 $\therefore 72=30+18+24$ .

PROP. II. THEOREM.

*Algeb.*—Let  $AB=a$ ,  $AC=x$ , and  $CB=y$ ;  
 then  $a=x+y$ ,  
 multiply both sides by  $a$ ,  
 $\therefore a^2=a(x+y)$ ,  
 $=ax+ay$ .  
 i.e.,  $AB^2=AB \cdot AC+AB \cdot BC$ .  
*Arith.*—Again, let  $a=9$ ,  $x=5$ , and  $y=4$ ;  
 then  $9=5+4$ ,  
 multiply both sides by 9,  
 $\therefore 9 \times 9=(9 \times 5)+(9 \times 4)$ ,  
 $9^2$  or  $81=45+36$ .

PROP. III. THEOREM.

*Algeb.*—Let  $AB=a$ ,  $BC=x$ , and  $AC=y$ ;  
 then  $a=x+y$ ,  
 multiply both sides by  $x$ ,  
 $\therefore ax=x(x+y)$ ,  
 $=x^2+xy$ .  
 i.e.,  $AB \cdot BC=AC \cdot CB+CB^2$ .  
*Arith.*—Again, let  $a=9$ ,  $x=6$ , and  $y=3$ ;  
 then  $9=6+3$ ,  
 multiply both sides by 6,  
 $\therefore 6 \times 9=6(6+3)$   
 $54=36+18$ .

PROP. IV. THEOREM.

*Algeb.*—Let  $AB=a$ ,  $AC=x$ , and  $BC=y$ ;  
 then  $a=x+y$ ,  
 squaring these equals,  $a^2=(x+y)^2$   
 $=x^2+2xy+y^2$ .  
 i.e.,  $AB^2=AC^2+CB^2+2AC \cdot CB$ .  
*Arith.*—Again, let  $a=12$ ,  $x=8$ , and  $y=4$ ;  
 then  $12=8+4$   
 squaring these equals,  $12^2=(8+4)^2$   
 $144=8^2+2(8 \times 4)+4^2$   
 $\therefore 144=64+64+16$ .

PROP. V. THEOREM.

*Algeb.*—Let  $AB=2a$ , then  $AC=CB=a$ ; and  $CD=x$ ;  
 then  $AD$ , the greater of the two unequal parts,  $=a+x$ ,  
 and  $DB$ , the less,  $=a-x$ ,  
 $\therefore (a+x)(a-x)=a^2-x^2$ ,  
 to each of these equals add  $x^2$ ,  
 $\therefore (a+x)(a-x)+x^2=a^2$ .  
 i.e.,  $AD \cdot DB+CD^2=CB^2$ .

*Arith.*—Again, let  $a=10$ , then  $2s=20$ ; and  $s=6$ ;  
 then  $AD=10+6$ , and  $DB=10-6$ ,  
 $(10+6)(10-6)=100-36$ ,  
 to each of these equals add  $6^2$  or  $36$ ,  
 $16 \times 4 + 36 = 100$ ,  
 $\therefore 64 + 36 = 100$ .

PROP. VI. THEOREM.

*Algeb.*—Let  $AB=2a$ , then  $AC=CB=a$ ; and  $CD=s$ ;  
 then  $AD$ , the greater of the two unequal parts,  $=s+a$ ,  
 and  $BD$ , the less,  $=s-a$ ,  
 $\therefore (s+a)(s-a)=s^2-a^2$ ,  
 by transposition,  $(s+a)(s-a)+a^2=s^2$ .  
 i.e.,  $AD \cdot DB+BC^2=CD^2$ .  
*Arith.*—Again, let  $a=8$ , then  $2s=16$ ; and  $s=12$ ;  
 then  $AD=12+8$ , and  $BD=12-8$ ,  
 $\therefore (12+8)(12-8)=12^2-8^2$ ,  
 by transposition,  $20 \times 4 + 8^2 = 12^2$ ,  
 $80 + 64 = 144$ .

PROP. VII. THEOREM.

*Algeb.*—Let  $AB=s$ ,  $AC=d$ , and  $CB=ny$ ;  
 then  $d=s-y$ ,  
 squaring these,  $d^2=s^2-2sy+y^2$ ,  
 by transposition,  $s^2+y^2=d^2+2sy$ ,  
 i.e.,  $AB^2+BC^2=2AB \cdot BC+AC^2$ .  
*Arith.*—Again, let  $s=16$ ,  $d=9$ , and  $y=7$ ;  
 then  $7=16-9$ ,  
 squaring these,  $7^2=16^2-2(9 \times 16)+9^2$ ,  
 $49=256-288+81$ ,  
 by transposition,  $49+288=256+81$ .

PROP. VIII. THEOREM.

*Algeb.*—Let  $AB=s$ , and  $CB=BD=y$ ;  
 then  $AB+BD=s+y$ ,  
 and  $AC$  or  $AB-BD=s-y$ ,  
 $(s+y)^2=s^2+2sy+y^2$ ,  
 $(s-y)^2=s^2-2sy+y^2$ ,  
 by subtraction,  $(s+y)^2-(s-y)^2=4sy$ ,  
 by transposition,  $(s+y)^2=4sy+(s-y)^2$ .  
 i.e.,  $4AB \cdot BC+AC^2=(AB+BC)^2$ .  
*Arith.*—Again, let  $a=16$ , and  $CB=BD=6$ ,  
 then  $AB+BD=16+6$ ,  
 and  $AB-BD=16-6$ ,  
 $(16+6)^2=256+192+100$ ,  
 $(16-6)^2=256-192+100$ ,  
 by subtraction,  $32^2-10^2=384$ ,  
 by transposition,  $484=384+100$ .

PROP. IX. THEOREM.

*Algeb.*—Let  $AC=CB=a$ , and  $CD=s$ ;  
 then  $AD$ , the greater of the two unequal parts,  $=a+s$ ,  
 and  $DB$ , the less,  $=a-s$ ,

$$\begin{aligned} (a+x)^2 &= a^2 + 2ax + x^2, \\ (a-x)^2 &= a^2 - 2ax + x^2, \\ \text{by addition,} \quad (a+x)^2 + (a-x)^2 &= 2a^2 + 2x^2, \\ \text{i.e., } AD^2 + DB^2 &= 2(AC^2 + CD^2). \end{aligned}$$

*Arith.*—Again, let  $a=10$ , and  $x=4$ ,

$$\begin{aligned} \text{then } AD &= 10+4 \\ \text{and } DB &= 10-4 \\ (10+4)^2 &= 100+80+16, \\ (10-4)^2 &= 100-80+16, \\ \text{by addition,} \quad 196+36 &= 200+32. \end{aligned}$$

PROP. X. THEOREM.

*Algeb.*—Let  $AC=CB=a$ , and  $CD=x$ ,

then  $AD$ , the greater of the two unequal parts,  $=x+a$ ,  
and  $BD$ , the less,

$$\begin{aligned} &= x-a, \\ (x+a)^2 &= x^2 + 2ax + a^2, \\ (x-a)^2 &= x^2 - 2ax + a^2, \\ \text{by addition,} \quad (x+a)^2 + (x-a)^2 &= 2x^2 + 2a^2, \\ \text{i.e., } AD^2 + DB^2 &= 2(AC^2 + CD^2). \end{aligned}$$

*Arith.*—Again, let  $a=10$ , and  $x=12$ ;

$$\begin{aligned} \text{then } AD &= 12+10, \text{ and } BD = 12-10, \\ (12+10)^2 &= 144+240+100 \\ (12-10)^2 &= 144-240+100 \\ \text{by addition,} \quad 484+4 &= 288+200. \end{aligned}$$

PROP. XI. THEOREM.

*Algeb.*—Let  $AB=a$ , and one of the unknown parts  $=x$ , the other will therefore  $=a-x$ ;

then by the problem

$$x^2 = a(a-x)$$

$$\therefore x^2 = a^2 - ax.$$

by transposition we have the quadratic equation,  $x^2 + ax = a^2$ ,

and completing the square,

$$x^2 + ax + \frac{a^2}{4} = a^2 + \frac{a^2}{4}$$

$$\left(x + \frac{a}{2}\right)^2 = \frac{5a^2}{4}$$

$$\therefore x + \frac{a}{2} = \frac{\pm a\sqrt{5}}{2}$$

$$\therefore x = \frac{\pm a\sqrt{5} - a}{2}.$$

Thus  $AH$  the one part  $= x = \frac{\sqrt{5}-1}{2} \cdot AB$ .

and  $HB$  the other part  $= a - x = \frac{3-\sqrt{5}}{2} \cdot AB$ .

*Arith.*—We cannot demonstrate this proposition *numerically*, for it is impossible to divide a number into two parts, so that the product of the whole number and one of the parts shall be *exactly* equal to the square of the other part. We can, however, approximate to the value of  $AH$  and  $HB$  by extending the root of 5 to any number of decimal places desired.

PROP. XII. THEOREM.

*Algeb.*—Let  $BC=a$ ,  $CA=b$ ,  $AB=c$ ,  $CD=x$ , and  $DA=y$ ;  
then  $BD=a+x$ ,

by Prop. 1. 47,

$$c^2 = (a+s)^2 + y^2,$$

$$\text{and } b^2 = s^2 + y^2,$$

by subtraction,

$$\therefore c^2 - b^2 = (a+s)^2 - s^2,$$

$$= a^2 + 2as + s^2 - s^2,$$

$$= a^2 + 2as,$$

by transposition,

$$\therefore c^2 = b^2 + a^2 + 2as,$$

$$\therefore c^2 \text{ is greater than } b^2 + a^2 \text{ by } 2as.$$

i.e.,  $AB^2$  is greater than  $AC^2 + CB^2$  by 2 BC . CD.

*Arith.*—Again, let  $a=5$ ,  $b=6708$ ,  $c=10$ ; and  $s=3$ , and  $y=6$ ;

then  $BD = a + s = 5 + 3$

by Prop. 1. 47,

$$10^2 = (5+3)^2 + 6^2,$$

$$6708^2 \text{ or } 45 = 3^2 + 6^2,$$

by subtraction,

$$10^2 - 45 = (5+3)^2 - 3^2 = 5^2 + 2(5 \times 3) + 3^2 - 3^2,$$

$$\therefore 10^2 = 45 + 5^2 + 2(5 \times 3),$$

i. e.,  $10^2$  is greater than  $45 + 5^2$  by  $2(5 \times 3)$ ,  
or 100 is greater than  $45 + 25$  by 80.

### PROP. XIII. THEOREM.

#### CASE I.

*Algeb.*—Let  $BC=a$ ,  $AC=b$ ,  $AB=c$ ; and  $BD=s$ , and  $AD=y$ .

Then  $DC=a-s$ ,

by Prop. 1. 47,

$$c^2 = y^2 + s^2,$$

$$b^2 = y^2 + (a-s)^2,$$

by subtraction,

$$\therefore c^2 - b^2 = s^2 - (a-s)^2,$$

$$= s^2 - a^2 + 2as - s^2,$$

$$= 2as - a^2,$$

by transposition,

$$a^2 + c^2 = b^2 + 2as,$$

$$\text{or } b^2 + 2as = a^2 + c^2,$$

$\therefore b^2$  is less than  $a^2 + c^2$  by  $2as$ .

i.e.,  $AC^2$  is less than  $CB^2 + BA^2$  by 2 CB . BD.

*Arith.*—Again, let  $a=10$ ,  $b=\sqrt{41}$  or 6'4031,  $c=\sqrt{61}$  or 7'8102; and  $s=6$ , and  $y=5$ ;

then  $DE = a - s = 10 - 6 = 4$ ,

by Prop. 1. 47,

$$7'8102^2 = 5^2 + 6^2, \text{ and } 6'4031^2 = 5^2 + 4^2,$$

by subtraction,  $7'8102^2 - 6'4031^2 = 6^2 - 4^2 = 6^2 - 10^2 + 2(10 \times 6) - 6^2 = 2(10 \times 6) - 10^2$ ,

by transposition,

$$10^2 + 7'8102^2 = 6'4031^2 + 2(10 \times 6),$$

$$\text{or } 6'4031^2 + 2(10 \times 6) = 10^2 + 7'8102^2,$$

i. e.  $6'4031^2$  is less than  $10^2 + 7'8102^2$  by  $2(10 \times 6)$ ,  
or 41 is less than  $100 + 61$  by 120.

#### CASE II.

*Algeb.*—

$DC = s - a$ ,

by Prop. 1. 47,

$$\therefore c^2 = s^2 + y^2,$$

$$b^2 = (s-a)^2 + y^2,$$

by subtraction,

$$c^2 - b^2 = s^2 - (s-a)^2,$$

$$= s^2 - s^2 + 2as - a^2,$$

$$= 2as - a^2,$$

by transposition,

$$a^2 + c^2 = b^2 + 2as,$$

$$\text{or } b^2 + 2as = a^2 + c^2,$$

$\therefore b^2$  is less than  $a^2 + c^2$  by  $2as$ .

i. e.  $AC^2$  is less than  $AB^2 + BC^2$  by 2 DE . BC.

*Arith.*—Again, let  $a=2$ ,  $b=\sqrt{41}$ ,  $c=\sqrt{61}$ ;  $x=6$ ,  $y=5$ ;  
 then  $DC=6-2=4$ ,  
 by Prop. 1. 47,  $61=6^2+5^2$ , and  $41=4^2+5^2$ ,  
 by subtraction,  $61-41=6^2-4^2=6^2-6^2+2(6\times 2)-2^2=2(6+2)-2^2$ ,  
 by transposition,  $2^2+61=41+2(6\times 2)$  or  $41+2(6\times 2)=2^2+61$ .  
 i. e. 41 is less than  $2^2+61$  by  $2(6\times 2)$ ,  
 or 41 is less than  $4+61$  by 24.

## CASE III.

*Algeb.*—Here AD and AC coincide, and also the points D and C;  
 by Prop. 1. 47,  $b^2+a^2=c^2$ ,  
 add  $a^2$  to both sides,  $b^2+2a^2=c^2+a^2$ ,  
 $\therefore b^2$  is less than  $c^2+a^2$  by  $2a^2$ , or  $2aa$ .  
 i. e.,  $AC^2$  is less than  $AB^2+BC^2$  by  $2BC^2$  or  $2BC \cdot BC$ .

*Arith.*—Again, let  $a=4$ ,  $b=5$ , and  $c=\sqrt{41}$ ;  
 by Prop. 1. 47,  $5^2+4^2=41$ ,  
 add  $4^2$ ,  $5^2+2\times 16=41+16$ ,  
 i. e.,  $5^2$  or 25 is less than  $41+16$  by  $2\times 16$

## PROP. XIV. THEOREM.

*Algeb.*—Let  $A=ab$ ,  $BE=b$ ,  $ED=a$ ;  $EH=x$ ;  
 then  $x^2=ab$ ,  
 $\therefore x=\sqrt{ab}$ .

*Arith.*—Again, let  $ab=36$ ,  $b=9$ , and  $c=4$ ;  
 then  $36=4\times 9$   
 $\therefore \sqrt{36}$  or  $6=s$ .

## GEOMETRICAL EXERCISES ON BOOK II.

1. The square of a line is four times the square of its half.
2. The rectangle under any two lines is equal to three times the rectangle under either of them, and a third of the other.
3. If two lines be divided, each into any number of parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the several parts of the one and the several parts of the other.
4. Prove Prop. II. 4 in a shorter way, by using some corollaries to the propositions in Book I.
5. Show that the difference of the squares of two unequal lines is equal to the rectangle contained by their sum and difference.
6. If a line CD be drawn from the vertex C of an isosceles triangle ABC to any point D in the base, or in the base produced, the difference of the squares of this line and one of the sides is equal to the rectangle contained by AD, DB.
7. The sum of the squares of two lines is never less than twice their rectangle.
8. The square of the sum of two lines is equal to four times the rectangle contained by them, together with the square of their difference.



9. The sum of the squares of two lines is equal to half the square of the sum, together with half the square of the difference.

10. ABC is a triangle, having the sides AB and AC equal: if AB is produced to D, so that BD is equal to AB, show that the square on CD is equal to the square on AB, together with twice the square on BC.

11. Show that in a parallelogram the squares of the diagonals are equal to the sum of the squares of all the sides.

12. The square on the base of an isosceles triangle, whose vertical angle is a right angle, is equal to four times the area of the triangle.

13. Any rectangle is the half of the rectangle contained by the diameters of the squares on its two sides.

14. If the points C, D, be equidistant from the extremities of the straight line AB, show that the squares constructed on AD and AC, exceed twice the rectangle AC, AD, by the square constructed on CD.

15. ABCD is a rectangle, E any point in BC, F in CD; show that the rectangle ABCD is equal to twice the triangle AEF, together with the rectangle BE, DF.

16. In any triangle ABC, if BP, CQ be drawn perpendicular to AC, AB, produced if necessary, then shall the square of BC be equal to the rectangle AB, BQ, together with the rectangle AC, CP.

17. If DE be drawn parallel to the base BC of an isosceles triangle ABC, then the square of BE is equal to the rectangle BC, DE, together with the square of CE.

18. If from any point within a rectangle lines be drawn to the angular points, the sums of the squares of those which are drawn to the opposite angles are equal.

19. If two sides of a trapezium be parallel to each other, the squares of its diagonals are together equal to the squares of its two sides, which are not parallel, and twice the rectangle contained by its parallel sides.

20. The squares of the diagonals of a trapezium are together double the squares of the two lines joining the bisections of the opposite sides.

21. ABC is a triangle in which C is a right angle, and DE is drawn from a point D in AC perpendicular to AB; show that the rectangle AB, AE is equal to the rectangle AC, AD.

22. In AB, the diameter of a circle, take two points C and D equally distant from the centre, and from any point E in the circumference draw EC, ED; show that the squares on EC and ED are together equal to the squares on AC and AD.

23. ABC is a triangle, of which the angle at C is obtuse, and the angle at B is half a right angle: D is the middle point of AB, and CE is drawn perpendicular to AB. Show that the square of AC is double of the squares of AD and DE.

24. Divide a given straight line into two parts, such that the squares of the whole line and of one of the parts shall be equal to twice the square of the other part.

25. In any triangle, if a line be drawn from the vertex bisecting the base, the sum of the squares of the two sides of the triangle is double the sum of the squares of the bisecting line and of half the base.

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# QUESTIONS AND EXERCISES

ON

## BOOK III.



### DEFINITIONS.

I. What are equal circles? II. When is a straight line said to touch a circle? III. When are circles said to touch one another? IV. When are straight lines said to be equally distant from the centre of a circle? V. When is one straight line said to be farther from the centre of a circle than another? VI. What is a segment of a circle? VII. What is the angle of a segment? VIII. What is an angle in a segment? IX. Upon what is an angle in a segment of a circle said to insist? X. What is a sector of a circle? XI. What are similar segments of circles?

### PROPOSITIONS AND COROLLARIES OF BOOK III.

Prop. 1. To find the centre of a given circle.

Cor. If in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

Prop. 2. If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Prop. 3. If a straight line, drawn through the centre of a circle, bisect a straight line in it which does not pass through the centre, it shall cut it at right angles: and conversely, if it cut it at right angles, it shall bisect it.

Prop. 4. If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

Prop. 5. If two circles cut one another, they shall not have the same centre.

Prop. 6. If one circle touch another internally, they shall not have the same centre.

Prop. 7. If any point be taken in the diameter of a circle, which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the line which passes through the centre is always greater than one more remote: and from the same point there can be drawn only two equal straight lines to the circumference, one upon each side of the diameter.

Prop. 8. If any point be taken without a circle, and straight lines be drawn

from it to the circumference, whereof one passes through the centre ; of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to that through the centre is always greater than one more remote : but of those which fall upon the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote: and only two equal straight lines can be drawn from the same point to the circumference, one upon each side of the line which passes through the centre.

Prop. 9. If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

Prop. 10. One circumference of a circle cannot cut another in more than two points.

Prop. 11. If one circle touch another internally in any point, the straight line which joins their centres, being produced, shall pass through that point of contact.

Prop. 12. If two circles touch each other externally in any point, the straight line which joins their centres shall pass through that point of contact.

Prop. 13. One circle cannot touch another in more points than one, whether it touch it on the inside or outside.

Prop. 14. Equal straight lines in a circle are equally distant from the centre; and those which are equally distant from the centre are equal to one another.

Prop. 15. The diameter is the greatest straight line in a circle ; and, of all others, that which is nearer to the centre is always greater than one more remote: and the greater is nearer to the centre than the less.

Prop. 16. The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle ; and no straight line can be drawn from the extremity between that straight line and the circumference, so as not to cut the circle; or, which is the same thing, no straight line can make so great an acute angle with the diameter at its extremity, or so small an angle with the straight line which is at right angles to it, as not to cut the circle.

Cor. The straight line which is drawn at right angles to the diameter of a circle from the extremity of it touches the circle; and it touches it only in one point; also there can be but one straight line which touches the circle in the same point.

Prop. 17. To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

Prop. 18. If a straight line touch a circle, the straight line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.

Prop. 19. If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Prop. 20. The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.

Prop. 21. The angles in the same segment of a circle are equal to one another.

Prop. 22. The opposite angles of any quadrilateral figure inscribed in a circle, are together equal to two right angles.

Prop. 23. Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

Prop. 24. Similar segments of circles, upon equal straight lines, are equal to one another.

Prop. 25. A segment of a circle being given, to describe the circle of which it is the segment.

Prop. 26. In equal circles, equal angles stand upon equal circumferences, whether they be at the centres or circumferences.

Prop. 27. In equal circles, the angles which stand upon equal circumferences are equal to one another, whether they be at the centres or circumferences.

Prop. 28. In equal circles, equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less.

Prop. 29. In equal circles, equal circumferences are subtended by equal straight lines.

Prop. 30. To bisect a given circumference, *that is, to divide it into two equal parts.*

Prop. 31. In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Cor. If the angle of a triangle be equal to the other two, it is a right angle, because the angle adjacent to it is equal to the same two, and when the adjacent angles are equal, they are right angles.

Prop. 32. If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle; the angles made by this line with the line touching the circle, shall be equal to the angles which are in the alternate segments of the circle.

Prop. 33. Upon a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

Prop. 34. From a given circle, to cut off a segment which shall contain an angle equal to a given rectilineal angle.

Prop. 35. If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

Prop. 36. If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.

Cor. If from any point without a circle there be drawn two straight lines cutting it, the rectangles contained by the whole lines and the parts of them without the circle are equal to one another.

Prop. 37. If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square of the line which meets it, the line which meets shall touch the circle.

## GEOMETRICAL EXERCISES ON BOOK III.

1. Describe a circle of given radius, which shall pass through two given points.

2. In the figure of Prop. III. 3, if any line be drawn parallel to AB, the diameter CD will bisect it.

3. Given two points in the circumference of a circle: describe the circle.

4. If two chords of a circle intersect each other, and make equal angles with the diameter passing through their point of contact, they are equal.

5. If one circle touch another internally, the distance between their centres is less than the difference of their radii.

6. Draw the shortest chord through a given point inside a circle.

7. If AD, CE be drawn perpendicular to the sides BC, AB of the triangle ABC, and DE be joined; prove that the angles ADE and ACE are equal to each other.

8. Parallel chords of a circle intercept equal arcs.

9. A quadrilateral is described so that its sides touch a circle: show that two of its sides are together equal to the other two sides.

10. Show that two tangents can be drawn to a circle from a given external point, and that they are of equal length.

NOTE.—A tangent is a straight line which touches a circle.

11. Right-angled triangles are described on the same hypotenuse; show that the angular points opposite the hypotenuse all lie on a circle described on the hypotenuse as diameter.

12. A chord PAQ cuts the diameter of a circle in A in an angle which is half a right angle; show that the squares of AP and AQ are together double of the square of the radius.

13. A is any point in the diameter (or diameter produced) of a circle, whose centre is O, OB a radius perpendicular to the diameter; if AB cut the circle in P, and the tangent in P cut AO in C, show that  $AC=OP$ .

14. If two circles touch each other, and parallel diameters be drawn, then lines which join the extremities of these diameters will pass through the point of contact.

15. ABCD is a parallelogram; draw CE perpendicular to the diagonal BD, and show that the perpendiculars upon AB, AD, at the points B, D, will intersect in CE.

16. The exterior angle of a quadrilateral figure inscribed in a circle, is equal to the interior and opposite.

17. Two circles intersect in A, B, the centre of one being in the circumference of the other; draw any chord ACD cutting them both; show that  $CB=CD$ .

18. If AB, CD, be chords of a circle at right angles to each other, prove that the sum of the arcs AC, BD, is equal to the sum of the arcs AD, BC.

19. With a given radius to describe a circle touching two given circles.

20. Two circles intersect in the points A and B; through A and B any two straight lines CAD, EBF, are drawn cutting the circles in the points C, D, E, F; prove that CE is parallel to DF.

21. If two isosceles triangles be of equal altitudes, and the side of one be equal to the side of the other their bases shall be equal.

22. If a straight line be drawn to touch a circle and parallel to a chord, the point of contact will be the middle point of the arc cut off by that chord.

23. If a circle be described on the radius of another circle, any straight line drawn from the point where they meet to the outer circumference, is bisected by the interior one.

24. If two opposite angles of a quadrilateral figure be together equal to two right angles, a circle may be described about it.

25. Of two circular segments upon the base, the larger is that which contains the smaller angle.

26. If two circles intersect, the common chord produced bisects the common tangent.

27. Four circles are described so that each one may touch internally three of the sides of a quadrilateral; show that a circle may be described so as to pass through the four centres of the above circles.

28. If two circles cut each other, the straight line joining their centres will bisect their common chord at right angles.

29. The perpendiculars let fall from the three angles of any triangle upon the opposite sides, intersect each other in the same point.

30. If a straight line touch the interior of two concentric circles and be placed in the exterior one, it will be bisected in the point of contact.

31. If from any two points in the circumference of the greater of two given concentric circles, two straight lines be drawn so as to touch the less circle, they shall be equal to one another.

32. If a quadrilateral rectilinear figure be described about a circle, the angles subtended, at the centre of the circle, by any two opposite sides of the figure, are, together, equal to two right angles.

33. If an arc of a circle be divided into three equal parts by three straight lines drawn from one extremity of the arc, the angle contained by two of the straight lines is bisected by the third.

34. If an equilateral triangle be inscribed in a circle, the square on a side thereof is equal to three times the square described upon the radius.

35. Given the vertical angle, the base, and the altitude of a triangle, to construct it.

36. Given the vertical angle, the base, and the sum of the sides of a triangle, to construct it.

37. Describe a circle which shall touch a given circle, and also touch a given line in a given point.

38. If on the three sides of any triangle, equilateral triangles be described; straight lines joining the centres of the circles described about these three triangles, will form an equilateral triangle.

39. ABD, ACE are two straight lines touching a circle at B and C, and if DE be joined, DE is equal to BD and CE together; show that DE touches the circle.

40. The greatest rectangle which can be inscribed in a circle is a square.

# QUESTIONS AND EXERCISES

ON

## BOOK IV.

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### DEFINITIONS.

I. When is a rectilinear figure said to be inscribed in another rectilinear figure? II. When is a rectilinear figure said to be described about another rectilinear figure? III. When is a rectilinear figure said to be inscribed in a circle? IV. When is a rectilinear figure said to be described about a circle? V. When is a circle said to be inscribed in a rectilinear figure? VI. When is a circle said to be described about a rectilinear figure? VII. When is a straight line said to be placed in a circle?

### PROPOSITIONS AND COROLLARIES OF BOOK IV.

Prop. 1. In a given circle, to place a straight line equal to a given straight line not greater than the diameter of a circle.

Prop. 2. In a given circle, to inscribe a triangle equiangular to a given triangle.

Prop. 3. About a given circle, to describe a triangle equiangular to a given triangle.

Prop. 4. To inscribe a circle in a given triangle.

Prop. 5. To describe a circle about a given triangle.

Cor. When the centre of a circle falls within the triangle, each of its angles is less than a right angle; but when the centre is in one of the sides of the triangle, the angle opposite to this side is a right angle; and if the centre falls without the triangle, the angle opposite to the side beyond which it is, is greater than a right angle. And conversely if the given triangle be acute-angled the centre of the circle falls within it; if it be a right-angled triangle the centre is in the side opposite to the right angle; and if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.

Prop. 6. To inscribe a square in a given circle.

Prop. 7. To describe a square about a given circle.

Prop. 8. To inscribe a circle in a given square.

Prop. 9. To describe a circle about a given square

Prop. 10. To describe an isosceles triangle, having each of the angles at the base double of the third angle.

Prop. 11. To inscribe an equilateral and equiangular pentagon in a given circle.

Prop. 12. To describe an equilateral and equiangular pentagon about a given circle.

Prop. 13. To inscribe a circle in a given equilateral and equiangular pentagon.

Prop. 14. To describe a circle about a given equilateral and equiangular pentagon.

Prop. 15. To inscribe an equilateral and equiangular hexagon in a given circle.

Cor. The side of the hexagon is equal to the straight line from the centre, that is, to the semi-diameter of the circle.

Prop. 16. To inscribe an equilateral and equiangular quindecagon in a given circle.

#### GEOMETRICAL EXERCISES ON BOOK IV.

1. In a given circle, to place a straight line equal and parallel to a straight line given in position, and not greater than a diameter.

2. In a given circle place a line of given length, which shall pass through a given point.

3. An equilateral triangle is inscribed in a circle, and through the angular points tangents are drawn; show that they will form an equilateral triangle, whose area is four times the former.

4. In figure Prop. iv. 4, show that the straight line DA will bisect the angle at A.

5. In figure Prop. iv. 5, show that the perpendicular from F on BC will bisect BC.

6. In the figure Prop. iv. 10, show that the angle A at the vertex of the triangle ABD is one-fifth of two right angles, and each of the angles at the base two-fifths of two right angles.

7. Divide a right angle into five equal parts.

8. In the figure Prop. iv. 10, show that AC is the side of a regular decagon inscribed in the larger circle.

9. In figure Prop. iv. 10, show that the angle ACD is equal to three times the angle at the vertex of the triangle.

10. On a given line to describe an equilateral and equiangular pentagon.

11. Given a regular pentagon; describe a triangle of the same area and altitude.

12. Describe an equilateral and equiangular octagon in a circle.

13. If two circles be described, one without and the other within a right-angled triangle, the sum of their diameters is equal to the sum of the sides containing the right angle.

14. Inscribe a square in a given right-angled isosceles triangle.

15. The centres of the inscribed and circumscribed circles of an equilateral triangle coincide, and the diameter of one is double that of the other.



16. The lines joining the alternate angles, or the intersections of the alternate sides of a regular pentagon, will form another regular pentagon.

17. Inscribe a circle in a given rhombus.

18. ABCDE is a regular pentagon; join AC and BE, and let BE meet AC in F; show that AC is equal to the sum of AB and BF.

19. The square inscribed in a circle is equal to half the square described about the same circle.

20. The centre of the circle which touches the two semicircles described on the sides of a right-angled triangle is the middle point of the hypotenuse.

21. A regular octagon inscribed in a circle is equal to the rectangle under the sides of the inscribed and circumscribing squares.

22. To inscribe a circle in a given quadrant.

23. The square on the side of a pentagon inscribed in a circle, is equal to the sum of the squares on the sides of a hexagon and decagon, inscribed in the same circle.

24. Describe a circle which shall pass through one angle, and touch two sides of a given square.

25. If DE be drawn parallel to the base BC of a triangle ABC, show that the circles described about the triangles ABC and ADE have a common tangent.

26. The angle ACB of any triangle is bisected, and the base AB is bisected at right angles, by straight lines which intersect at D; show that the angles ACB, ADB are together equal to two right angles.

27. If ABCDEF is a regular hexagon, and AC, BD, CE, DF, EA, FB, be joined, another hexagon will be formed whose area is one-third of that of the former.

28. If any number of parallelograms be inscribed in a given parallelogram, the diameters of all the figures shall cut one another in the same point.

29. If ABCDE be any pentagon inscribed in a circle, and AC, BD, CE, DA, EB, be joined, then are the angles ABE, BCA, CDB, DEC, EAD, together equal to two right angles.

30. If in any circle the side of an inscribed hexagon be produced till it becomes equal to the side of an inscribed square, a tangent drawn from the extremity, without the circle, shall be equal to the side of an inscribed octagon.

# QUESTIONS AND EXERCISES

ON

## BOOK V.

### DEFINITIONS.

I. When is a less magnitude said to be a part of a greater? II. A greater magnitude a multiple of a less? III. What is ratio? IV. When are magnitudes said to have a ratio to one another? V. Of four magnitudes, when is the first said to have to the second the same ratio which the third has to the fourth? VI. What are proportionals? VII. Of four magnitudes, when is the first said to have a greater ratio to the second than the third to the fourth? VIII. What is 'analogy' or 'proportion'? IX. Of how many terms must a proportion consist? X. If three magnitudes are proportionals, what is the relation between the first and second as compared with that between the first and third? XI. If four magnitudes are continued proportionals, what is the relation between the first and second as compared with the relation between the first and fourth? Define 'compound ratio.' XII. What is meant by 'homologous' terms in proportionals? XIII. What is meant by the terms 'permuto' or 'alternando' as used in proportion? XIV. 'Invertendo'? XV. 'Componendo'? XVI. 'Dividendo'? XVII. 'Convertendo'? XVIII. Explain the term, '*Ex æquali distantia*' or '*Ex æquo*.' XIX. '*Ex æquali*.' XX. '*Ex æquali in proportione perturbatâ*.'

### AXIOMS.

I. What is the relation between equimultiples of the same or of equal magnitudes? II. What is the relation between magnitudes of which the same or equal magnitudes are equimultiples? III. The relation between a multiple of a greater magnitude and the same multiple of the less? IV. How are magnitudes related, a multiple of one of which is greater than the same multiple of another?

### PROPOSITIONS OF BOOK V.

Prop. 1. If any number of magnitudes be equimultiples of as many, each of each; what multiple soever any one of them is of its part, the same multiple shall all the first magnitudes be of all the others?

Prop. 2. If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the

sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.

Prop. 3. If the first be the same multiple of the second, which the third is of the fourth; and if of the first and third there be taken equimultiples; these shall be equimultiples, the one of the second, and the other of the fourth.

Prop. 4. If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz. 'the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.'

Prop. 5. If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder shall be the same multiple of the remainder, that the whole is of the whole.

Prop. 6. If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two; the remainders are either equal to these others, or equimultiples of them.

Prop. A. If the first of four magnitudes has the same ratio to the second, which the third has to the fourth; then, if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.

Prop. B. If four magnitudes are proportionals, they are proportionals also when taken inversely.

Prop. C. If the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second, as the third is to the fourth.

Prop. D. If the first be to the second as is the third to the fourth, and if the first be a multiple, or a part of the second; the third is the same multiple, or the same part of the fourth.

Prop. 7. Equal magnitudes have the same ratio to the same magnitude: and the same has the same ratio to equal magnitudes.

Prop. 8. Of two unequal magnitudes, the greater has a greater ratio to any other magnitude than the less has: and the same magnitude has a greater ratio to the less of two other magnitudes, than it has to the greater.

Prop. 9. Magnitudes which have the same ratio to the same magnitude are equal to one another: and those to which the same magnitude has the same ratio are equal to one another.

Prop. 10. That magnitude which has a greater ratio than another has unto the same magnitude, is the greater of the two: and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the less of the two.

Prop. 11. Ratios that are the same to the same ratio, are the same to one another.

Prop. 12. If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

Prop. 13. If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth has to the sixth.

Prop. 14. If the first has the same ratio to the second which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

Prop. 15. Magnitudes have the same ratio to one another which their equimultiples have.

Prop. 16. If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

Prop. 17. If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately: that is, if the first be to the second, as the third to the fourth, the first and second together shall be to the third, as the third and fourth together to the fourth.

Prop. 18. If magnitudes, taken separately, be proportionals, they shall also be proportionals when taken jointly: that is, if the first be to the second, as the third to the fourth, the first and second together shall be to the third, as the third and fourth together to the fourth.

Prop. 19. If a whole magnitude be to a whole, as a magnitude taken from the first is to a magnitude taken from the other; the remainder shall be to the remainder as the whole to the whole.

Prop. E. If four magnitudes be proportionals, they are also proportionals by conversion; that is, the first is to its excess above the second, as the third to its excess above the fourth.

Prop. 20. If there be three magnitudes, and other three, which, taken two and two, have the same ratio; then, if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Prop. 21. If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; then if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Prop. 22. If there be any number of magnitudes, and as many others, which taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes, the same ratio which the first has to the last of the others. N.B. This is usually cited by the words 'ex æquali,' or 'ex æquo.'

Prop. 23. If there be any number of magnitudes, and as many others, which taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first has to the last of the others. N.B. This is usually cited by the words 'ex æquali in proportionibus perturbatâ;' or 'ex æquo perturbato.'

Prop. 24. If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.

Prop. 25. If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.

Prop. F. Ratios which are compounded of the same ratios, are the same to one another.

Prop. G. If several ratios be the same to several ratios, each to each; the ratio which is compounded of ratios which are the same to the first ratios, each to each, shall be the same to the ratio compounded of ratios which are the same to the other ratios each to each.

Prop. H. If a ratio which is compounded of several ratios be the same to a ratio which is compounded of several other ratios; and if one of the first ratios, or the ratio which is compounded of several of them, be the same to one of the last ratios, or to the ratio which is compounded of several of them; then the remaining ratio of the first, or, if there be more than one, the ratio compounded of the remaining ratios, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio compounded of these remaining ratios.

Prop. K. If there be any number of ratios, and any number of other ratios such, that the ratio which is compounded of ratios which are the same to the first ratios, each to each, is the same to the ratio which is compounded of ratios which are the same, each to each, to the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same to several of the first ratios, each to each, be the same to one of the last ratios, or to the ratio which is compounded of ratios which are the same, each to each, to several of the last ratios; then the remaining ratio of the first, or, if there be more than one, the ratio which is compounded of ratios which are the same each to each to the remaining ratios of the first, shall be the same to the remaining ratio of the last or, if there be more than one, to the ratio which is compounded of ratios which are the same each to each to these remaining ratios.

### THE FIFTH BOOK OF EUCLID PROVED ALGEBRAICALLY.

The Fifth Book is read chiefly by advanced students for the sake of its logic. With young students, perhaps, the better plan is to follow the recommendation of De Morgan—either reject the book altogether and substitute the numerical definition of proportion, or if the book be retained make the numerical definition accompanying it; for ratio which is essentially a property of number, will be more readily apprehended by arithmetic, than by the tedious but correct method of Euclid.

It is well, in the first place, to observe the distinction between the geometrical and the arithmetical view of the subject. In geometry the question is confined to the equality or inequality of lines, angles, surfaces, and solids, without reference to any unit whereby they are estimated. Geometry thus gives a definite, but at the same time, an indeterminate measure. Arithmetic, on the other hand, gives or 'attempts to give the exact values in numbers.' A geometrical ratio can therefore only be represented by the magnitudes themselves; thus, the ratio of two lines is expressed by the length of the lines themselves. But an arithmetical ratio may be represented by a third term, which will be an exact number when the terms of the ratio are commensurable, and an approximation to it when they are incommensurable. This distinction seems to have been recognised by Euclid, who has treated of arithmetical ratio in Books VII., VIII., and IX., whilst in Book V. he has given a definition the widest and most general of proportion, purposely avoiding, no doubt, any reference whatever to the exact numerical measure of ratios.

#### DEFINITIONS.

1. If  $a$  and  $b$  be two quantities, and  $m$  be any integer which indicates the number of times  $b$  is contained in  $a$ , that is  $a=mb$ , then  $b$  is a *part* or submultiple of  $a$ .
2. If  $a=mb$ ,  $m$  as before being any integer,  $a$  is said to be a multiple of  $b$ .

3. The quantity of a ratio is known by dividing the antecedent by the consequent ; as the ratio of  $a : b$  is expressed by  $\frac{a}{b}$ .

5. The following will show the meaning of this definition :—Let  $a, b, x, y$  be four magnitudes, then the first  $a$  is said to have to the second  $b$ , the same ratio which the third  $x$  has to the fourth  $y$ , when if  $a$  and  $x$  are multiplied by any number whatever as  $m$ , and  $b$  and  $y$  are multiplied by any other number as  $n$  it is found that—

i. If  $ma$  be greater than  $nb$ , then  $mx$  is greater than  $ny$ .

ii. If  $ma = nb$ , then  $mx = ny$ .

iii. If  $ma$  be less than  $nb$ , then  $mx$  is less than  $ny$ .

The second case is true only when the terms are commensurable, the other two cases apply to magnitudes which are incommensurable.\*

6. If the ratio of  $a : b$  be the same as the ratio of  $x : y$ , the four terms form a proportion, and is thus expressed—

$$a : b :: x : y,$$

which is read  $a$  is to  $b$  as  $x$  to  $y$ .

7. Let the four magnitudes  $a, b, x, y$  be taken as in def. 5 ; that is,  $a : b :: x : y$ , and let  $ma$  be greater than  $nb$ , but  $mx$  not greater than  $ny$  ; then  $a$  is said to have to  $b$  a greater ratio than  $x$  has to  $y$ , and  $x$  is said to have to  $y$  a less ratio than  $a$  has to  $b$ .

10. If  $a : b :: b : c$ ,  $a$  is said to have to  $c$  the duplicate ratio of that which it has to  $b$ .

11. If  $a : b :: b : c :: c : d :: d : e$ , &c.,  $a$  is said to have to  $d$  the triplicate ratio of that which it has to  $b$ , and to  $e$  the quadruplicate ratio of that which it has to  $b$ , &c.

12. If  $a : b :: x : y$  ;  $a$  and  $x$  the antecedents are said to be homologous to one another ; and  $b$  and  $y$  the consequents are homologous.

13. If  $a : b :: x : y$ , 'permutando,' or 'alternando'  $a : x :: b : y$ .

14. If  $a : b :: x : y$ , 'invertendo'  $b : a :: y : x$ .

15. If  $a : b :: x : y$ , 'componendo'  $a + b : b :: x + y : y$ .

16. If  $a : b :: x : y$ , 'dividendo'  $a - b : b :: x - y : y$  ; or which is the same thing (v. 17), if  $a + b : b :: x + y : y$ ,  $a : b :: x : y$ .

17. If  $a : b :: x : y$ , 'convertendo'  $a : a - b :: x : x - y$ .

19. Let the first series of magnitudes be

$$a, b, c, \dots \quad l, m, n ;$$

and let the second be,

$$p, q, r, \dots \quad x, y, z.$$

\* It has sometimes been objected to this definition that it bears no resemblance to the common notions of ratios employed in Arithmetic and Algebra. The truth is, this resemblance seems never to have been intended. By comparing, however, the Arithmetical definition with Euclid's, we shall see that they both lead to the same result. The obscurity so often complained of in this definition, arises from the very general and wide terms which it is necessary to employ in order to include every case of equimultiples, not simply such multiples as the doubles and trebles, but every conceivable multiple.

$$\text{If } a : b :: p : q,$$

$$b : c :: q : r,$$

$$\&c. \quad \&c.$$

$$l : m :: x : y,$$

$$m : n :: y : z.$$

$$\text{'Ex aequali' } a : n :: p : z.$$

20. Let the first series of magnitudes be

$$a, b, c \dots \dots l, m, n;$$

and let the second be,

$$p, q, r \dots \dots x, y, z.$$

$$\text{If } a : b :: y : z,$$

$$b : c :: x : y,$$

$$\&c. \quad \&c.$$

$$l : m :: q : r,$$

$$m : n :: p : q$$

$$\text{'Ex aequali in proportione perturbatâ,'}$$

$$a : n :: p : x.$$

#### PROP. I.

Let each of the magnitudes  $a, b, c, \&c.$ , be equimultiples of as many  $p, q, r, \&c.$ ; so that  $a = mp, b = mq, c = mr, \&c.$

Then shall  $a + b + c + \&c. = m(p + q + r + \&c.)$

$$\text{For } \because a = mp, \quad b = mq, \quad c = mr, \&c.$$

$$\therefore a + b + c + \&c. = mp + mq + mr, \&c.$$

$$\therefore a + b + c + \&c. = m(p + q + r + \&c.)$$

That is  $a + b + c + \&c.$  is the same multiple of  $p + q + r + \&c.$ , that each of  $a, b, c, \&c$  is of its part  $p, q, r, \&c.$

#### PROP. II.

Let  $a$  the first magnitude, be the same multiple of  $b$  the second, that  $c$  the third is of  $d$  the fourth; and  $e$  the fifth the same multiple of  $b$  the second, that  $f$  the sixth is of  $d$  the fourth; so that,

$$a = mb, \quad c = nd,$$

$$e = mb, \quad f = nd.$$

Then shall  $a + e$  be the same multiple of  $b$  that  $c + f$  is of  $d$ .

$$\text{For } \because a = mb, \quad \text{and } e = mb,$$

$$\therefore a + e = mb + mb = (m + n) b.$$

by addition

$$\therefore c = md, \quad \text{and } f = nd,$$

Similarly

$$\therefore c + f = md + nd = (m + n) d.$$

It has been shown that  $a + e$  contains  $b, (m + n)$  times, and  $c + f$  also contains  $d, (m + n)$  times; therefore  $a + e$  is the same multiple of  $b$  that  $c + f$  is of  $d$ .

#### PROP. III.

Let  $a, b, x, y$ , be four magnitudes such that  $a$  is the same multiple of  $b$  that  $x$  is of  $y$ , and let  $c, z$ , be equimultiples of  $a$  and  $x$ ; that is, let  $a = mb, x = my$ , and  $c = na, z = nx$ .

Then shall  $c, z$  be also equimultiples of  $b$  and  $y$ .

$$\text{For } \because a = mb, \quad \text{and } c = na,$$

$$\therefore c = n \times mb = nmb.$$

$$\text{And } \because x = my, \quad \text{and } z = nx,$$

$$\therefore z = n \times my = nmy;$$

It has been shown that  $c$ , and  $z$  contain  $b$  and  $y$  respectively  $nm$  times;

$\therefore c, z$  are equimultiples of  $b$  and  $y$ .

PROP. IV.

Let  $a, b, x, y$  be proportionals, that is, let  $a : b :: x : y$ .

Then  $ma : nb :: mx : ny$ .

For  $\because a : b :: x : y$ ,

$$\therefore \frac{a}{b} = \frac{x}{y};$$

multiply these equals by  $\frac{m}{n}$ ,  $m$  and  $n$  being any integers,

$$\therefore \frac{ma}{nb} = \frac{mx}{ny};$$

that is,  $ma : nb :: mx : ny$ .

Cor. If  $n$  be unity, then  $ma : b :: mx : y$ ;

and if  $m$  be unity, then  $a : nb :: n : ny$ .

PROP. V.

Let  $a$  be the same multiple of  $b$ , that  $x$  is of  $y$ , so that  $a = mb$ ,  $x = my$ ,  $x$  and  $y$  being parts of  $a$  and  $b$ .

Then shall  $a - x = m(b - y)$ .

For  $\because a = mb$ , and  $x = my$ ,

by subtraction,

$$a - x = mb - my,$$

$$\therefore a - x = m(b - y),$$

that is,  $a - x$  is the same multiple of  $b - y$  that  $a$  is of  $b$ .

PROP. VI.

Let  $a$  and  $b$  be equimultiples of  $x$  and  $y$ , and let  $c$  and  $d$  subtracted from  $a$  and  $b$ , also be equimultiples of  $x$  and  $y$ ; so that  $a = mx$ ,  $b = my$ , and  $c = nx$ ,  $d = ny$ .

Then shall  $a - c$  and  $b - d$  be either equal to  $x$  and  $y$ , or equimultiples of them.

For  $\because a = mx$ , and  $c = nx$ ,

by subtraction,

$$a - c = mx - nx = (m - n)x;$$

and  $\because b = my$ , and  $d = ny$ ,

by subtraction,

$$b - d = my - ny = (m - n)y;$$

If  $m - n = 1$ , then  $a - c = x$ , and  $b - d = y$ , but if not, then  $a - c$ , and  $b - d$  contain  $x$  and  $y$ ,  $(m - n)$  times;

that is,  $a - c$  and  $b - d$  are either equal to  $x$  and  $y$ , or are equimultiples of them.

PROP. A.

Let  $a, b, x, y$  be four proportionals, that is  $a : b :: x : y$ .

Then if  $a$  be greater than  $b$ ,  $x$  shall be also greater than  $y$ ; and if equal, equal; and if less, less.

For  $\because a : b :: x : y$ ;

$$\therefore \frac{a}{b} = \frac{x}{y};$$

(1) if, then,  $a$  be greater than  $b$ ,  $x$  must be greater than  $y$ ;

(2) if  $a$  be equal to  $b$ ,  $x$  must be equal to  $y$ ;

(3) if  $a$  be less than  $b$ ,  $x$  must be also less than  $y$ .

PROP. B. 'INVERTENDO.'

Let  $a, b, x, y$  be proportionals, that is,  $a : b :: x : y$ .

Then shall  $b : a :: y : x$ .

For  $\because a : b :: x : y$ ,

$$\therefore \frac{a}{b} = \frac{x}{y};$$

and if 1 be divided by each of these equals,

$$1 \div \frac{a}{b} = 1 \div \frac{x}{y},$$

$$\therefore 1 \times \frac{b}{a} = 1 \times \frac{y}{x},$$

$$\text{i.e. } \frac{b}{a} = \frac{y}{x},$$

$$\therefore b : a :: y : x.$$



PROP. C.

Let  $a, b, x, y$  be four magnitudes ;

First, let  $a=mb, x=my$ ,

Then shall  $a:b::x:y$ .

$$\text{For } \because a=mb, \quad \therefore \frac{a}{b}=m;$$

$$\text{and } \because x=my, \quad \therefore \frac{x}{y}=m;$$

$$\text{hence } \therefore \frac{a}{b}=\frac{x}{y},$$

$$\therefore a:b::x:y.$$

Secondly, let  $a=\frac{b}{n}, x=\frac{y}{n}$ .

Then shall  $a:b::x:y$ .

$$\text{For } \because a=\frac{b}{n}, \quad \therefore \frac{a}{b}=\frac{1}{n};$$

$$\text{and } \because x=\frac{y}{n}, \quad \therefore \frac{x}{y}=\frac{1}{n};$$

$$\text{hence } \therefore \frac{a}{b}=\frac{x}{y},$$

$$\therefore a:b::x:y.$$

PROP. D.

Let  $a, b, x, y$  be proportionals, that is,  $a:b::x:y$ .

First, let  $a$  be a multiple of  $b$ ; so that,  $a=mb$ .

Then shall  $x=my$ .

For since  $a:b::x:y$ .

$$\therefore \frac{a}{b}=\frac{x}{y}$$

$$\text{and } \because a=mb, \quad \therefore \frac{a}{b}=m;$$

$$\therefore m=\frac{x}{y},$$

$$\text{i.e. } x=my;$$

therefore  $x$  is the same multiple of  $y$  that  $a$  is of  $b$ .

Secondly, let  $a$  be a part of  $b$ ; so that,  $a=\frac{b}{n}$ .

Then shall  $x=\frac{y}{n}$ .

$$\text{For since } \frac{a}{b}=\frac{x}{y}; \quad \text{and } a=\frac{b}{n},$$

$$\therefore \frac{a}{b}=\frac{1}{n};$$

$$\therefore \frac{x}{y}=\frac{1}{n},$$

$$\therefore x=\frac{y}{n},$$

therefore  $x$  is the same part of  $y$  that  $a$  is of  $b$ .

PROPS. VII., VIII., IX.

Are so obvious as not to require algebraical proof.

PROP. X.

First, let  $a$  have a greater ratio to  $c$  than  $b$  has to  $c$ .

Then shall  $a$  be greater than  $b$ .

Since  $a$  has a greater ratio to  $c$  than  $b$  has to  $c$ ,

$$\therefore \frac{a}{c} \text{ is greater than } \frac{b}{c},$$

$$\therefore a \text{ is greater than } b.$$

Secondly, let  $c$  have a greater ratio to  $b$  than  $c$  has to  $a$ ;

Then shall  $b$  be less than  $a$ .

Since  $c$  has a greater ratio to  $b$  than  $c$  has to  $a$ ,

$$\therefore \frac{c}{b} \text{ is greater than } \frac{c}{a};$$

$$\therefore b \text{ is less than } a.$$

#### PROP. XI.

Let the ratio of  $a : b$  be the same as the ratio of  $c : d$ , and the ratio of  $c : d$  the same as the ratio of  $e : f$ .

Then shall the ratio of  $a : b$  be the same as the ratio of  $e : f$ .

For  $\because a : b :: c : d$ ,

$$\therefore \frac{a}{b} = \frac{c}{d};$$

and  $\because c : d = e : f$ ,

$$\therefore \frac{c}{d} = \frac{e}{f};$$

$$\text{hence } \frac{a}{b} = \frac{e}{f};$$

$$\text{and } a : b :: e : f.$$

#### PROP. XII.

Let  $a, b, c, d, e, f$ , be proportionals, so that  $a : b :: c : d :: e : f$ .

Then shall  $a : b :: a + c + e : b + d + f$ .

For  $\because a : b :: c : d :: e : f$ ,

$$\therefore \frac{a}{b} = \frac{c}{d} = \frac{e}{f};$$

$$\text{and } \because \frac{a}{b} = \frac{c}{d},$$

$$\therefore ad = bc,$$

$$\therefore \frac{a}{b} = \frac{e}{f},$$

$$\therefore af = be,$$

$$\text{also } ab = ba,$$

$$\text{hence } a(b + d + f) = b(a + c + e),$$

dividing these equals by  $b(b + d + f)$ ,

$$\therefore \frac{a}{b} = \frac{a + c + e}{b + d + f};$$

$$\text{and } a : b :: a + c + e : b + d + f.$$

#### PROP. XIII.

Let  $a, b, c, d, e, f$ , be six magnitudes, so that  $a : b :: c : d$ , but that  $c$  has a greater ratio to  $d$  than  $c$  has to  $f$ .

Then shall  $a$  have a greater ratio to  $b$  than  $e$  has to  $f$ .

For  $\because a : b :: c : d$ ,

$$\therefore \frac{a}{b} = \frac{c}{d};$$

but since  $c$  has a greater ratio to  $d$  than  $c$  has to  $f$ ,

$$\therefore \frac{c}{d} \text{ is greater than } \frac{c}{f},$$

$$\text{also } \therefore \frac{a}{b} \text{ is greater than } \frac{c}{f}.$$

that is, the ratio of  $a$  to  $b$  is greater than the ratio of  $e$  to  $f$ .

#### PROP. XIV.

Let  $a, b, x, y$ , be proportionals, that is  $a : b :: x : y$ .

Then if  $a$  be greater than  $a$ ,  $b$  shall be greater than  $y$ ; and if equal, equal; and if less, less.

For  $\because a : b :: x : y$ ,

$$\therefore \frac{a}{b} = \frac{x}{y},$$

multiply these equals by  $\frac{b}{x}$ ,

$$\therefore \frac{a}{x} = \frac{b}{y};$$

but if  $a$  be greater than  $x$ , then must  $b$  be greater than  $y$ ,  
for if it were not so, the fraction  $\frac{b}{y}$  could not be equal to  $\frac{a}{x}$ ;

$\therefore b$  is greater than  $y$ .

In the same manner, if  $a = x$ ,

$\therefore b$  is equal to  $y$ .

And if  $a$  be less than  $x$ ,

$\therefore b$  is less than  $y$ .

#### PROP. XV.

Let  $a, b$ , be any magnitudes of the same kind, and let  $x$  and  $y$  be equimultiples of  $a$  and  $b$ ; so that  $x = ma$ , and  $y = mb$ .

Then shall  $a : b :: x : y$ .

For  $\because x = ma$ , and  $y = mb$ ,

by division,

$$\therefore \frac{x}{y} = \frac{ma}{mb};$$

$$\therefore \frac{x}{y} = \frac{a}{b} \text{ or } \frac{a}{b} = \frac{x}{y}$$

and  $\therefore a : b :: x : y$ .

#### PROP. XVI. 'PERMUTANDO' OR 'ALTERNANDO.'

Let  $a, b, x, y$  be four magnitudes of the same kind, which are proportionals  
viz.  $a : b :: x : y$ .

Then shall  $a : x :: b : y$ .

For  $\because a : b :: x : y$ ,

$$\therefore \frac{a}{b} = \frac{x}{y};$$

multiply these equals by  $\frac{b}{x}$ ,

$$\therefore \frac{ab}{bx} = \frac{xb}{yx},$$

$$\therefore \frac{a}{x} = \frac{b}{y},$$

and  $\therefore a : x :: b : y$ .

#### PROP. XVII. 'DIVIDENDO.'

Let  $a+b, b, x+y, y$ , be proportionals, that is,  $a+b : b :: x+y : y$ .

Then shall  $a : b :: x : y$ .

For  $\because a+b : b :: x+y : y$ ,

$$\therefore \frac{a+b}{b} = \frac{x+y}{y},$$

$$\therefore \frac{a}{b} + 1 = \frac{x}{y} + 1,$$

and taking 1 from each of these equals,

$$\therefore \frac{a}{b} = \frac{x}{y},$$

and  $a : b :: x : y$ .

#### PROP. XVIII. 'COMPONENDO.'

Let  $a : b :: x : y$ .

Then shall  $a+b : b :: x+y : y$ .

For  $\because a : b :: x : y$ ,

$$\therefore \frac{a}{b} = \frac{x}{y};$$

and adding 1 to each of these equals,

$$\therefore \frac{a}{b} + 1 = \frac{x}{y} + 1$$

$$\therefore \frac{a+b}{b} = \frac{x+y}{y}$$

$$\text{and } a+b : b :: x+y : y.$$

PROP. XIX.

Let the whole  $a$  have the same ratio to the whole  $b$ , that the part  $x$  taken from the first, has to the part  $y$  taken from the second; that is, let  $a : b :: x : y$ .

Then shall  $a-x : b-y :: a : b$ .

For  $\therefore a : b :: x : y$ ,

$\therefore a : x :: b : y$ ,

$\therefore a-x : x :: b-y : y$ ,

$\therefore a-x : b-y :: x : y$ ,

$a : b :: x : y$ ,

alternando,

dividendo,

alternando,

but by hypothesis,

and ratios that are the same to the same ratio are equal to one another (v. 11);

$$\therefore a-x : b-y :: a : b.$$

PROP. E. 'CONVERTENDO.'

Let  $a : b :: x : y$ .

Then shall  $a : a-b :: x : x-y$ .

For  $\therefore a : b :: x : y$ ,

$$\therefore \frac{a}{b} = \frac{x}{y},$$

dividendo,

$$\therefore a-b : b :: x-y : y,$$

invertendo,

$$\therefore b : a-b :: y : x-y,$$

componendo,

$$\therefore a : a-b :: x : x-y.$$

PROP. XX.

Let  $a, b, c$ , be three magnitudes, and  $x, y, z$ , other three; so that  $a : b :: x : y$ , and  $b : c :: y : z$ .

Then if  $a$  be greater than  $c$ ,  $x$  shall be greater than  $z$ ; and if equal, equal; and if less, less.

$$\text{For } \therefore a : b :: x : y, \quad \therefore \frac{a}{b} = \frac{x}{y};$$

$$\text{and } \therefore b : c :: y : z, \quad \therefore \frac{b}{c} = \frac{y}{z};$$

and multiplying these equals,

$$\therefore \frac{ab}{bc} = \frac{xy}{yz},$$

$$\text{or } \frac{a}{c} = \frac{x}{z}.$$

And if  $a$  be greater than  $c$ ,  $x$  must be also greater than  $z$ ,

for if it were not so, the fraction  $\frac{a}{c}$  could not be equal to  $\frac{x}{z}$ ;

$\therefore x$  is greater than  $z$ .

And in the same way it may be shown,

if  $a$  be equal to  $c$ ,  $x$  is equal to  $z$ ,

and if  $a$  be less than  $c$ ,  $x$  is less than  $z$ .

PROP. XXI.

This may be proved similarly to the preceding one.

## PROP. XXII. 'EX ÆQUALI' OR 'EX ÆQUO.'

Let  $a, b, c$  be three magnitudes,  $p, q, r$  other three, so that  $a : b :: p : q$ , and  $b : c :: q : r$ .  
Then shall  $a : c :: p : r$ .

$$\text{For } \because a : b :: p : q, \quad \therefore \frac{a}{b} = \frac{p}{q},$$

$$\text{and } \because b : c :: q : r, \quad \therefore \frac{b}{c} = \frac{q}{r},$$

multiplying these equals,

$$\therefore \frac{a}{b} \times \frac{b}{c} = \frac{p}{q} \times \frac{q}{r},$$

$$\text{or } \frac{a}{c} = \frac{p}{r}$$

$$\text{and } \therefore a : c :: p : r.$$

The same proof may be extended to four or any number of magnitudes.

## PROP. XXIII. 'EX ÆQUALI IN PROPORTIONE PERTURBATO' OR 'EX ÆQUO PERTURBATÂ.'

Let  $a, b, c$  be three magnitudes, and  $p, q, r$  other three, so that  $a : b :: q : r$ , and  $b : c :: p : q$ .

Then shall  $a : c :: p : r$ .

$$\text{For } \because a : b :: q : r, \quad \therefore \frac{a}{b} = \frac{q}{r};$$

$$\text{and } \because b : c :: p : q, \quad \therefore \frac{b}{c} = \frac{p}{q};$$

multiplying these equals,

$$\therefore \frac{a}{b} \times \frac{b}{c} = \frac{q}{r} \times \frac{p}{q},$$

$$\text{or } \frac{a}{c} = \frac{p}{r}.$$

$$\text{and } \therefore a : c :: p : r.$$

The same proof may be extended to four or any number of magnitudes.

## PROP. XXIV,

Let  $a : b :: c : d$ , and  $e : b :: f : d$ .

Then shall  $a + e : b :: c + f : d$ .

$$\text{For } \because a : b :: c : d, \quad \therefore \frac{a}{b} = \frac{c}{d};$$

$$\text{and } \because e : b :: f : d, \quad \therefore \frac{e}{b} = \frac{f}{d};$$

by addition,

$$\frac{a}{b} + \frac{e}{b} = \frac{c}{d} + \frac{f}{d},$$

$$\therefore \frac{a+e}{b} = \frac{c+f}{d},$$

$$\text{and } \therefore a+e : b :: c+f : d.$$

Cor. 1. Similarly it may be shown that

$$a-e : b :: c-f : d.$$

## PROP. XXV.

Let  $a : b :: x : y$ ; and let  $a$  be the greatest of the four, and consequently  $d$  least. (V. 14 and A.)

Then shall  $a+y$  be greater than  $b+x$ .

$$\text{For } \because a : b :: x : y,$$

$$\therefore a-b : b :: x-y : y,$$

$$\therefore b : a-b :: y : x-y,$$

but  $b$  is greater than  $y$  (hyp.),

also  $\therefore a-b$  is greater than  $x-y$  (V. 14);

add to each of these unequals  $b+y$ ;

$$\therefore a-b+b+y \text{ is greater than } x-y+b+y,$$

$$\text{or } a+y \text{ is greater than } b+x.$$

## GEOMETRICAL EXERCISES ON BOOK V.

1. If the first of four proportional magnitudes be greater than the second, the third is also greater than the fourth; if equal, equal; and if less, less.

2. If the first of four magnitudes be the same multiple of the second, or the same part of it, that the third is of the fourth, the first is to the second, as the third is to the fourth.

3. If the first of four proportional magnitudes be a multiple, or a part, of the second, the third is the same multiple, or the same part, of the fourth.

4. If any number of equal ratios be each greater than a given ratio, the ratio of the sum of their antecedents to the sum of their consequents, shall be greater than that given ratio.

5. If the first of four magnitudes have a greater ratio to the second than the third has to the fourth, the second shall have to the first a less-ratio than the fourth has to the third.

6. If the first of four magnitudes, of the same kind, have a greater ratio to the second than the third has to the fourth, the first shall have to the third a greater ratio than the second has to the fourth.

7. If four magnitudes of the same kind be proportionals, and if the first of them be the greatest, the fourth shall be the least; but if the first of them be the least, the fourth shall be the greatest.

8. If the first, together with the second, of four magnitudes, have a greater ratio to the second, than the third, together with the fourth, has to the fourth, the first shall have a greater ratio to the second than the third has to the fourth.

9. If the first of four magnitudes have a greater ratio to the second than the third has to the fourth, the first, together with the second, shall have to the second a greater ratio than the third, together with the fourth, has to the fifth.

10. If the first term of a ratio be less than the second, the ratio shall be increased by adding the same quantity to both terms, but if the first term be greater than the second, the ratio shall be diminished by adding the same quantity to both.

11. If the first of four magnitudes, of the same kind, have a greater ratio to the second than the third has to the fourth, the first, together with the third, shall have to the second, together with the fourth, a greater ratio than the third has to the fourth, and a less ratio than the first has to the second.

12. If the first, together with the second, have to the second a greater ratio than the third, together with the fourth, has to the fourth, then shall the first, together with the second, have to the first a less ratio than the third, together with the fourth, has to the third.

13. If the first, together with the second have to the third, together with the fourth, a greater ratio than the first has to the third, then shall the second have to the fourth a greater ratio than the first, together with the second, has to the third, together with the fourth.

14. If any number of magnitudes be continual proportionals, their differences shall also be proportional.

15. If there be three magnitudes, and other three, and if the first have a greater ratio to the second, in the former set, than the first has to the second, in the latter; and if, also, the second have to the third, in the former set, a greater ratio than the second has to the third, in the latter; then shall the first have a greater ratio to the third, in the former set, than the first has to the third, in the latter.

# QUESTIONS AND EXERCISES

ON

## BOOK VI.

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### DEFINITIONS.

I. What are similar rectilineal figures? II. What are reciprocal triangles and parallelograms? III. When is a straight line said to be cut in extreme and mean ratio? What is meant by the altitude of any figure?

Prop. 1. Triangles and parallelograms of the same altitude are one to the other as their bases.

*Cor.* Triangles and parallelograms that have equal altitudes are to one another as their bases.

Prop. 2. If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or these produced, proportionally: and conversely, if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

Prop. 3. If the angle of a triangle be divided into two equal angles, by a straight line which also cuts the base; the segments of the base shall have the same ratio which the other sides of the triangle have to one another: and conversely, if the segments of the base have the same ratio which the other sides of the triangle have to one another; the straight line drawn from the vertex to the point of section, divides the vertical angle into two equal angles.

Prop. A. If the outward angle of a triangle made by producing one of its sides, be divided into two equal angles, by a straight line, which also cuts the base produced; the segments between the dividing line and the extremities of the base, have the same ratio which the other sides of the triangle have to one another; and conversely, if the segments of the base produced have the same ratio which the other sides of the triangle have; the straight line drawn from the vertex to the point of section divides the outward angle of the triangle into two equal angles.

Prop. 4. The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.

Prop. 5. If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular; and the equal angles shall be those which are opposite to the homologous sides.

Prop. 6. If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

Prop. 7. If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals; then, if each of the remaining angles be either less, or not less, than a right angle, or if one of them be a right angle; the triangles shall be equiangular, and shall have those angles equal about which the sides are proportionals.

Prop. 8. In a right-angled triangle, if a perpendicular be drawn from the right angle to the base; the triangles on each side of it are similar to the whole triangle, and to one another.

*Cor.* The perpendicular drawn from the right angle of a right-angled triangle to the base, is a mean proportional between the segments of the base.

Prop. 9. From a given straight line to cut off any part required.

Prop. 10. To divide a given straight line similarly to a given divided straight line, that is, into parts that shall have the same ratios to one another which the parts of the divided given straight line have.

Prop. 11. To find a third proportional to two given straight lines.

Prop. 12. To find a fourth proportional to three given straight lines.

Prop. 13. To find a mean proportional between two given straight lines.

Prop. 14. Equal parallelograms, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and conversely, parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Prop. 15. Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and conversely, triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Prop. 16. If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: and conversely, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

Prop. 17. If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean; and conversely, if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are proportionals.

Prop. 18. Upon a given straight line to describe a rectilineal figure similar, and similarly situated, to a given rectilineal figure.

Prop. 19. Similar triangles are to one another in the duplicate ratio of their homologous sides.

*Cor.* If three straight lines be proportionals as the first is to the third, so is any triangle upon the first to a similar, and similarly described triangle upon the second.

Prop. 20. Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have.



*Cor. 1.* Similar rectilinear figures of any number of sides, are to one another in the duplicate ratio of their homologous sides.

*Cor. 2.* If three straight lines be proportionals, as the first is to the third, so is any rectilinear figure upon the first, to a similar and similarly described rectilinear figure upon the second.

Prop. 21. Rectilinear figures which are similar to the same rectilinear figure, are also similar to one another.

Prop. 22. If four straight lines be proportionals, the similar rectilinear figures similarly described upon them shall also be proportionals: and conversely, if the similar rectilinear figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.

Prop. 23. Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

Prop. 24. Parallelograms about the diameter of any parallelogram, are similar to the whole, and to one another.

Prop. 25. To describe a rectilinear figure which shall be similar to one, and equal to another given rectilinear figure.

Prop. 26. If two similar parallelograms have a common angle, and be similarly situated; they are about the same diameter.

Prop. 27. Of all parallelograms applied to the same straight line, and deficient by parallelograms, similar and similarly situated to that which is described upon the half of the line; that which is applied to the half, and is similar to its defect, is the greatest.

Prop. 28. To a given straight line to apply a parallelogram equal to a given rectilinear figure, and deficient by a parallelogram similar to a given parallelogram: but the given rectilinear figure to which the parallelogram to be applied is to be equal, must not be greater than the parallelogram applied to half of the given line, having its defect similar to the defect of that which is to be applied; that is, to the given parallelogram.

Prop. 29. To a given straight line to apply a parallelogram equal to a given rectilinear figure, exceeding by a parallelogram similar to another given.

Prop. 30. To cut a given straight line in extreme and mean ratio.

Prop. 31. In right-angled triangles, the rectilinear figure described upon the side opposite to the right angle, is equal to the similar and similarly described figures upon the sides containing the right angle.

Prop. 32. If two triangles which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another; the remaining sides shall be in a straight line.

Prop. 33. In equal circles, angles, whether at the centres or circumferences, have the same ratio which the circumferences on which they stand have to one another: so also have the sectors.

Prop. B. If an angle of a triangle be bisected by a straight line which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line which bisects the angle.

Prop. C. If from any angle of a triangle, a straight line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described *about the triangle*.

Prop. D. The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

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### GEOMETRICAL EXERCISES ON BOOK VI.

1. If the bases of four rectangles be proportionals, and their altitudes be also proportionals, the rectangles themselves shall likewise be proportionals.

2. Prove that in iv. 10, one of the triangles is a mean proportional between the other two.

3. From a point E in the common base of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD, meeting BC, BD at F, G; prove that FG is parallel to CD.

4. The rectangle contained by two lines is a mean proportional between the squares.

5. The diameter of a circle is a mean proportional between the sides of an equilateral triangle and hexagon about the circle.

6. Equiangular parallelograms have to one another the same ratio as the rectangles contained by the sides about equal angles in each.

7. In a triangle ABC a straight line AD is drawn perpendicular to the straight line BD which bisects the angle B; show that a straight line drawn from D parallel to BC will bisect AC.

8. AB is a diameter of a circle, CD is a chord at right angles to it, and E is any point in CD; AE and BE are drawn and produced to cut the circle at F and G; show that the quadrilateral CFDG has any two of its adjacent sides in the same ratio as the remaining two.

9. If ABC be an inscribed triangle, and BD be drawn parallel to the tangent at A to meet AC, or AC produced, show that AB is a mean proportional between AC, AD.

10. If the diagonals of an inscribed quadrilateral intersect at right angles, the rectangles of the opposite sides are together double of the area of the figure.

11. Inscribe in a given triangle a parallelogram similar to a given parallelogram.

12. The line which cuts at equal angles the lines containing a given angle is the least that can be drawn to cut off a triangle of given area.

13. The straight lines, drawn from the bisections of the three sides of a triangle to the opposite angles meet in the same point.

14. If two trapeziums have an angle of the one equal to an angle of the other, and if, also, the sides of the two figures, about each of their angles, be proportionals, the remaining angles of the one shall be equal to the remaining angles of the other.

15. To divide a given finite straight line into two parts, the squares of which shall be to one another in a given ratio.

16. The base, the vertical angle, and the ratio of the two sides of a triangle being given, to construct it.

17. In a given circle place a straight line parallel to a given straight line, and having a given ratio to it; the ratio not being greater than that of the diameter to the given line in the circle.

19. ABC is an isosceles triangle; draw AD perpendicular to the base, and DDE, cutting AB, AC, in E, F; then  $AD : DE :: AB + AF : AB - AF$ .

20. In any triangle, inscribe a triangle similar to a given triangle.

21. In a given equilateral and equiangular pentagon to inscribe a square.

22. Similar triangles, and similar polygons, are to one another as any rectilinear figure described upon any side of the one is to a similar rectilinear figure similarly described upon the homologous side of the other.

23. To cut off from a given parallelogram a similar parallelogram, which shall be any required part of it.

24. Any rectangle is the half of the rectangle contained by the diameters of the squares of its two sides.

25. In different circles the radii which bound equal sectors contain angles reciprocally proportional to their circles; and conversely.

26. In the figure of VI. 24 show that EG and KH are parallel.

27. A circle is described round an equilateral triangle, and from any point in the circumference straight lines are drawn to the angular points of the triangle; show that one of these straight lines is equal to the other two together.

28. If a rectangle be inscribed in a right-angled triangle, having the right angle common, the rectangle of the segments of the hypotenuse will be equal to the sum of the rectangles of the segments of the sides.

29. Bisect a triangle by a line drawn parallel to one of its sides.

30. Describe a rhombus, equal to a given rectilinear figure, and having an angle equal to a given rectilinear angle.

31. Through a given point draw a line so that, if perpendiculars be dropped upon it from two other given points, the parts intercepted between the given point and the feet of the perpendiculars may be equal.

32. If three circles touch each other, two of which are equal, the vertical angle of the triangle formed by joining the points of contact is equal to the angle at the base of the isosceles triangle formed by joining the three centres.

33. A square is inscribed in a right-angled triangle ABC, one side DE of the square coinciding with the hypotenuse AB of the triangle: show that the area of the square is equal to the rectangle AD, BE.

34. Any regular polygon inscribed in a circle is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides.

35. To make an isosceles triangle which shall be equal to a scalene triangle, and shall also have an equal vertical angle with it.

# QUESTIONS AND EXERCISES

ON

## BOOK XI.



i. What is a solid? ii. What is that which bounds a solid? iii. When is a straight line perpendicular to a plane? iv. When is a plane perpendicular to a plane? v. How is the inclination of a straight line to a plane measured? vi. What is the inclination of a plane to a plane? vii. When are two planes said to have the same inclination to each other which two other planes have? viii. What are parallel planes? ix. What is a solid angle? x. What are similar solid figures? xi. A pyramid? xii. A prism? xiii. A sphere? xiv. What is the axis of a sphere? xv. The centre of a sphere? xvi. The diameter of a sphere? xvii. Define a cone and distinguish between a right-angled, an obtuse-angled, and an acute-angled cone. What is the axis of a cone? The base of a cone? Define a cylinder? What is the axis of a cylinder? What are the bases of a cylinder? What are similar cones and cylinders? What is a cube? Define a tetrahedron. An octahedron. A dodecahedron. An icesahedron. What is a parallelopiped?

Prop. 1. One part of a straight line cannot be in a plane, and another part above it.

Prop. 2. Two straight lines which cut one another are in one plane, and three straight lines which meet one another are in one plane.

Prop. 3. If two planes cut one another, their common section is a straight line.

Prop. 4. If a straight line stand at right angles to each of two straight lines in the point of their intersection, it shall also be at right angles to the plane which passes through them, that is, to the plane in which they are.

Prop. 5. If three straight lines meet all in one point, and a straight line stands at right angles to each of them in that point; these three straight lines are in one and the same plane.

Prop. 6. If two straight lines be at right angles to the same plane, they shall be parallel to one another.

Prop. 7. If two straight lines be parallel, the straight line drawn from any point in the one to any point in the other, is in the same plane with the parallels.

Prop. 8. If two straight lines be parallel, and one of them is at right angles to a plane; the other also shall be at right angles to the same plane.

Prop. 9. Two straight lines which are each of them parallel to the same straight line, and not in the same plane with it, are parallel to one another.

Prop. 10. If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the first two; the first two and the other two shall contain equal angles.

Prop. 11. To draw a straight line perpendicular to a plane, from a given point above it.

Prop. 12. To erect a straight line at right angles to a given plane, from a point given in the plane.

Prop. 13. From the same point in a given plane, there cannot be two straight lines at right angles to the plane, upon the same side of it; and there can be but one perpendicular to a plane from a point above the plane.

Prop. 14. Planes to which the same straight line is perpendicular, are parallel to one another.

Prop. 15. If two straight lines meeting one another, be parallel to two other straight lines which meet one another, but are not in the same plane with the first two; the plane which passes through these is parallel to the plane passing through the others.

Prop. 16. If two parallel planes be cut by another plane, their common sections with it are parallel.

Prop. 17. If two straight lines be cut by parallel planes, they shall be cut in the same ratio.

Prop. 18. If a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.

Prop. 19. If two planes cutting one another be each of them perpendicular to a third plane; their common section shall be perpendicular to the same plane.

Prop. 20. If a solid angle be contained by three plane angles, any two of them are greater than the third.

Prop. 21. Every solid angle is contained by plane angles, which together are less than four right angles.

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### EXERCISES ON BOOK XI.

1. If P be any point in the plane of a parallelogram ABCD, show that the triangle PBD is equal to the sum of the triangles PAB, PBC.

2. Draw a perpendicular in two given lines, not in the same plane.

3. Two planes being perpendicular to each other, draw a third, perpendicular to both.

4. If two planes which are not parallel be cut by two other parallel planes, the lines of section of the first by the last two will contain equal angles.

5. If two planes intersect, and straight lines are drawn in one of the planes from a point in the common intersection making equal angles with it, show that *they are equally inclined to the other plane.*

6. From a point A a perpendicular is drawn to a plane meeting it at B; from B a perpendicular is drawn on a straight line in the plane meeting it at C; show that AC is perpendicular to the straight line in the plane.

7. From a point E draw EC, ED perpendicular to the plane CAB, DAB which intersect in AB, and from D draw DF perpendicular to the plane CAB meeting it at F; show that the straight line CF, produced if necessary, is perpendicular to AB.

8. Show that the perpendicular drawn from the vertex of a regular tetrahedron on the opposite face is three times that drawn from its own foot on any of the other faces.

9. Having three points given in a plane, find a point above the plane equidistant from either of them.

10. Two points are taken on two walls, which meet at an angle; find the shortest line which joins them.

11. There can be only five regular solids.

12. Two of the three plane angles which form a solid angle, and also the inclination of their planes, being given, it is required to find the third plane angle.

13. Polygons formed by cutting a prism by parallel planes are equal.

14. Polygons formed by cutting a pyramid by parallel planes are similar.

15. Of all the angles, which a straight line makes with any straight lines drawn in a given plane to meet it, the least is that which measures the inclination of the line to the plane.

16. ABC is a triangle; the perpendiculars from A and B on the opposite sides meet at D; through D a straight line is drawn perpendicular to the plane of the triangle, and E is any point in this straight line; show that the straight line joining E to any angular point of the triangle is at right angles to the straight line drawn through that angular point parallel to the opposite side of the triangle.

17. Perpendiculars are drawn from a point to a plane, and to a straight line in that plane; show that the straight line joining the feet of the perpendiculars is perpendicular to the former straight line.

18. From two given points, draw equal lines to the same point of a given line, not in the same plane with them.

19. If two lines are parallel, the common section of any two planes passing through them will be parallel to either.

20. If a pyramid with a polygon for its base be cut by a plane parallel to the base, the section will be a polygon similar to the base.

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